THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XVII.
[MAY, 1887—APRIL, 1888.]

MACMILLAN AND CO.
London and Cambridge.
1888.

1887, Nov. 10 – 1888, April 16.
Haven fund.

CAMBRIDGE:
PRINTED BY W. METCALFE AND SON, TRINITY STREET AND ROSE CRESCENT.

CONTENTS.

ARITHMETIC, ALGEBRA, AND TRIGONOMETRY.

GEOMETRY OF TWO AND THREE DIMENSIONS.

DIFFERENTIAL AND INTEGRAL CALCULUS AND DIFFERENTIAL EQUATIONS.

THEORY OF ELLIPTIC FUNCTIONS.

APPLIED MATHEMATICS.

No. CXCIII.] NEW SERIES. [May, 1887.

THE MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, M.A., F.R.S.,

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XVII.—NO. 1.

MACMILLAN AND CO.

London and Cambridge.

1887.

W. METCALFE AND SON, } Price—One Shilling. { PRINTERS, CAMBRIDGE.

MESSENGER OF MATHEMATICS.

ON THE TRANSFORMATION AND DEVELOPMENTS OF THE TWELVE ELLIPTIC FUNCTIONS AND THE FOUR ZETA FUNCTIONS.

By *J. W. L. Glaisher.*

THE principal object of this paper is to give the transformations of the elliptic and Zeta functions, which are due to the change of q into $-q$, q^2 and $q^{\frac{1}{2}}$; and also the first few terms of the expansions of these functions in ascending powers of the arguments. I omit the demonstrations, giving only the results in a form convenient for reference.

This paper (in so far as it relates to the elliptic functions) may be regarded as a continuation of two papers "On Elliptic Functions," which were published in vol. XI. of the *Messenger*, (pp. 81–95, 120–138).

In §§ 17–19 I have given tables of the values of the elliptic functions when the argument is increased by K, iK' or $K+iK'$, and also the values of the functions for certain special values of the argument. I have found these tables of such constant use in working with the twelve elliptic functions that I have been tempted to include them in this paper. The first table in § 17 was given in vol. XI., p. 88, but the functions were there arranged in an inconvenient order.

For the sake of completeness the q-series for the elliptic and Zeta functions are also given (§§ 2, 3).

Notation, § 1.

§ 1. The letters ρ and u are used to denote

$$\frac{2K}{\pi} \text{ and } \frac{2Kx}{\pi}$$

respectively; so that $u=\rho x$. The argument x is supposed to be independent of k. The letters h and h' are used to denote k^2 and k'^2 respectively.

The q-series for $k\rho$ sn u, &c., and $\rho Z(u)$, § 2.

§ 2. The q-series for the twelve elliptic functions are:*

$$k\rho \operatorname{sn} u = \Sigma_1^\infty \frac{4q^{\frac{1}{2}(2n-1)}}{1-q^{2n-1}} \sin(2n-1)x,$$

$$k\rho \operatorname{cn} u = \Sigma_1^\infty \frac{4q^{\frac{1}{2}(2n-1)}}{1+q^{2n-1}} \cos(2n-1)x,$$

$$\rho \operatorname{dn} u = \Sigma_1^\infty \frac{4q^n}{1+q^{2n}} \cos 2nx;$$

$$\rho \operatorname{ns} u = \frac{1}{\sin x} + \Sigma_1^\infty \frac{4q^{2n-1}}{1-q^{2n-1}} \sin(2n-1)x,$$

$$\rho \operatorname{ds} u = \frac{1}{\sin x} - \Sigma_1^\infty \frac{4q^{2n-1}}{1+q^{2n-1}} \sin(2n-1)x,$$

$$\rho \operatorname{cs} u = \frac{\cos x}{\sin x} - \Sigma_1^\infty \frac{4q^{2n}}{1+q^{2n}} \sin 2nx;$$

$$\rho \operatorname{dc} u = \frac{1}{\cos x} + \Sigma_1^\infty (-)^{n-1} \frac{4q^{2n-1}}{1-q^{2n-1}} \cos(2n-1)x,$$

$$k'\rho \operatorname{nc} u = \frac{1}{\cos x} - \Sigma_1^\infty (-)^{n-1} \frac{4q^{2n-1}}{1+q^{2n-1}} \cos(2n-1)x,$$

$$k'\rho \operatorname{sc} u = \frac{\sin x}{\cos x} - \Sigma_1^\infty (-)^{n-1} \frac{4q^{2n}}{1+q^{2n}} \sin 2nx;$$

$$k\rho \operatorname{cd} u = \Sigma_1^\infty (-)^{n-1} \frac{4q^{\frac{1}{2}(2n-1)}}{1-q^{2n-1}} \cos(2n-1)x,$$

$$kk'\rho \operatorname{sd} u = \Sigma_1^\infty (-)^{n-1} \frac{4q^{\frac{1}{2}(2n-1)}}{1+q^{2n-1}} \sin(2n-1)x,$$

$$k'\rho \operatorname{nd} u = 1 - \Sigma_1^\infty (-)^{n-1} \frac{4q^n}{1+q^{2n}} \cos 2nx;$$

---

* *Messenger*, vol. XVI., pp. 137, 138.

and those for the four Zeta functions are :*

$$\rho Z(u) = \qquad \Sigma_1^\infty \frac{4q^n}{1-q^{2n}} \sin 2nx,$$

$$\rho Z_1(u) = \frac{\cos x}{\sin x} + \Sigma_1^\infty \frac{4q^{2n}}{1-q^{2n}} \sin 2nx,$$

$$\rho Z_2(u) = -\frac{\sin x}{\cos x} - \Sigma_1^\infty (-)^{n-1} \frac{4q^{2n}}{1-q^{2n}} \sin 2nx,$$

$$\rho Z_3(u) = \qquad - \Sigma_1^\infty (-)^{n-1} \frac{4q^n}{1-q^{2n}} \sin 2nx.$$

Transformations of $k\rho \operatorname{sn} u$, &c., §§ 3, 4.

§ 3. The changes in the functions $k\rho \operatorname{sn} u$, &c. which are due to the change of q into $-q$ are as follows:

q	$-q$,
$k\rho \operatorname{sn} u$,	$ikk'\rho \operatorname{sd} u$,
$k\rho \operatorname{cn} u$,	$ik\rho \operatorname{cd} u$,
$\rho \operatorname{dn} u$,	$k'\rho \operatorname{nd} u$;
$\rho \operatorname{ns} u$,	$\rho \operatorname{ds} u$,
$\rho \operatorname{ds} u$,	$\rho \operatorname{ns} u$,
$\rho \operatorname{cs} u$,	$\rho \operatorname{cs} u$;
$\rho \operatorname{dc} u$,	$k'\rho \operatorname{nc} u$,
$k'\rho \operatorname{nc} u$,	$\rho \operatorname{dc} u$,
$k\rho \operatorname{sc} u$,	$k'\rho \operatorname{sc} u$;
$k\rho \operatorname{cd} u$,	$ik\rho \operatorname{cn} u$,
$kk'\rho \operatorname{sd} u$,	$ik\rho \operatorname{sn} u$,
$k'\rho \operatorname{nd} u$,	$\rho \operatorname{dn} u$.

The changes in the functions which are due to the change of q into q^2 and $q^{\frac{1}{2}}$ are shown in the table on the next page:

† *Messenger*, vol. xv., p. 145.

q	q^2	$q^{\frac{1}{2}}$
$k\rho\ \mathrm{sn}\,u$	$\tfrac{1}{2}k^2\rho\,\dfrac{\mathrm{sn}\,\tfrac{1}{2}u\ \mathrm{cn}\,\tfrac{1}{2}u}{\mathrm{dn}\,\tfrac{1}{2}u}$	$2k^{\frac{1}{2}}\rho\,\dfrac{(1+k)\,\mathrm{sn}\,u}{1+k\,\mathrm{sn}^2u}$
$k\rho\ \mathrm{cn}\,u$	$\tfrac{1}{2}\rho\,\dfrac{\mathrm{dn}^2\tfrac{1}{2}u-k'}{\mathrm{dn}\,\tfrac{1}{2}u}$	$2k^{\frac{1}{2}}\rho\,\dfrac{\mathrm{cn}\,u\ \mathrm{dn}\,u}{1+k\,\mathrm{sn}^2u}$
$\rho\ \mathrm{dn}\,u$	$\tfrac{1}{2}\rho\,\dfrac{\mathrm{dn}^2\tfrac{1}{2}u+k'}{\mathrm{dn}\,\tfrac{1}{2}u}$	$(1+k)\,\rho\,\dfrac{1-k\,\mathrm{sn}^2u}{1+k\,\mathrm{sn}^2u}$
$\rho\ \mathrm{ns}\,u$	$\tfrac{1}{2}\rho\,\dfrac{\mathrm{dn}\,\tfrac{1}{2}u}{\mathrm{sn}\,\tfrac{1}{2}u\ \mathrm{cn}\,\tfrac{1}{2}u}$	$\rho\,\dfrac{1+k\,\mathrm{sn}^2u}{\mathrm{sn}\,u}$
$\rho\ \mathrm{ds}\,u$	$\tfrac{1}{2}\rho\,\dfrac{\mathrm{cn}^2\tfrac{1}{2}u+k'\,\mathrm{sn}^2\tfrac{1}{2}u}{\mathrm{sn}\,\tfrac{1}{2}u\ \mathrm{cn}\,\tfrac{1}{2}u}$	$\rho\,\dfrac{1-k\,\mathrm{sn}^2u}{\mathrm{sn}\,u}$
$\rho\ \mathrm{cs}\,u$	$\tfrac{1}{2}\rho\,\dfrac{\mathrm{cn}^2\tfrac{1}{2}u-k'\,\mathrm{sn}^2\tfrac{1}{2}u}{\mathrm{sn}\,\tfrac{1}{2}u\ \mathrm{cn}\,\tfrac{1}{2}u}$	$\rho\,\dfrac{\mathrm{cn}\,u\ \mathrm{dn}\,u}{\mathrm{sn}\,u}$
$\rho\ \mathrm{dc}\,u$	$\tfrac{1}{2}\rho\,\dfrac{\mathrm{dn}^2\tfrac{1}{2}u+k'}{\mathrm{cn}^2\tfrac{1}{2}u-k'\,\mathrm{sn}^2\tfrac{1}{2}u}$	$(1+k)\,\rho\,\dfrac{1-k\,\mathrm{sn}^2u}{\mathrm{cn}\,u\ \mathrm{dn}\,u}$
$k'\rho\ \mathrm{nc}\,u$	$k'^{\frac{1}{2}}\rho\,\dfrac{\mathrm{dn}\,\tfrac{1}{2}u}{\mathrm{cn}^2\tfrac{1}{2}u-k'\,\mathrm{sn}^2\tfrac{1}{2}u}$	$(1-k)\,\rho\,\dfrac{1+k\,\mathrm{sn}^2u}{\mathrm{cn}\,u\ \mathrm{dn}\,u}$
$k'\rho\ \mathrm{sc}\,u$	$k'^{\frac{1}{2}}\rho\,\dfrac{(1+k')\,\mathrm{sn}\,\tfrac{1}{2}u\ \mathrm{cn}\,\tfrac{1}{2}u}{\mathrm{cn}^2\tfrac{1}{2}u-k'\,\mathrm{sn}^2\tfrac{1}{2}u}$	$k'^2\rho\,\dfrac{\mathrm{sn}\,u}{\mathrm{cn}\,u\ \mathrm{dn}\,u}$
$k\rho\ \mathrm{cd}\,u$	$\tfrac{1}{2}\rho\,\dfrac{\mathrm{dn}^2\tfrac{1}{2}u-k'}{\mathrm{cn}^2\tfrac{1}{2}u+k'\,\mathrm{sn}^2\tfrac{1}{2}u}$	$2k^{\frac{1}{2}}\rho\,\dfrac{\mathrm{cn}\,u\ \mathrm{dn}\,u}{1-k\,\mathrm{sn}^2u}$
$kk'\rho\ \mathrm{sd}\,u$	$k'^{\frac{1}{2}}\rho\,\dfrac{(1-k')\,\mathrm{sn}\,\tfrac{1}{2}u\ \mathrm{cn}\,\tfrac{1}{2}u}{\mathrm{cn}^2\tfrac{1}{2}u+k'\,\mathrm{sn}^2\tfrac{1}{2}u}$	$2k^{\frac{1}{2}}\rho\,\dfrac{(1-k)\,\mathrm{sn}\,u}{1-k\,\mathrm{sn}^2u}$
$k'\rho\ \mathrm{nd}\,u$	$k'^{\frac{1}{2}}\rho\,\dfrac{\mathrm{dn}\,\tfrac{1}{2}u}{\mathrm{cn}^2\tfrac{1}{2}u+k'\,\mathrm{sn}^2\tfrac{1}{2}u}$	$(1-k)\,\rho\,\dfrac{1+k\,\mathrm{sn}^2u}{1-k\,\mathrm{sn}^2u}$

The quantities $\mathrm{dn}^2\tfrac{1}{2}u \pm k'$ which occur in the first column of results may be expressed in the forms

$$(1 \pm k')\,(\mathrm{cn}^2\tfrac{1}{2}u \pm k'\,\mathrm{sn}^2\tfrac{1}{2}u),$$

and the quantities $(1 \pm k)\,(1 \mp k\,\mathrm{sn}^2u)$ which occur in the second column may be expressed in the forms

$$\mathrm{dn}^2u \pm k\,\mathrm{cn}^2u.$$

§ 4. In the following cases the transformed quantities are expressible very simply in terms of sums or differences of elliptic or Zeta functions.

q	q^2	$q^{\frac{1}{2}}$
$k\rho\ \mathrm{sn}\,u$	$\tfrac{1}{2}\rho\,\{Z(\tfrac{1}{2}u) - Z_3(\tfrac{1}{2}u)\}$	
$k\rho\ \mathrm{cn}\,u$	$\tfrac{1}{2}\rho\,\{\mathrm{dn}\,\tfrac{1}{2}u - k'\,\mathrm{nd}\,\tfrac{1}{2}u\}$	
$\rho\ \mathrm{dn}\,u$	$\tfrac{1}{2}\rho\,\{\mathrm{dn}\,\tfrac{1}{2}u + k'\,\mathrm{nd}\,\tfrac{1}{2}u\}$	
$\rho\ \mathrm{ns}\,u$	$\tfrac{1}{2}\rho\,\{Z_1(\tfrac{1}{2}u) - Z_2(\tfrac{1}{2}u)\}$	$\rho\,\{\mathrm{ns}\,u + k\,\mathrm{sn}\,u\}$
$\rho\ \mathrm{ds}\,u$	$\tfrac{1}{2}\rho\,\{\mathrm{cs}\,\tfrac{1}{2}u + k'\,\mathrm{sc}\,\tfrac{1}{2}u\}$	$\rho\,\{\mathrm{ns}\,u - k\,\mathrm{sn}\,u\}$
$\rho\ \mathrm{cs}\,u$	$\tfrac{1}{2}\rho\,\{\mathrm{cs}\,\tfrac{1}{2}u - k'\,\mathrm{sc}\,\tfrac{1}{2}u\}$	$\rho\,\{Z_1(u) - Z(u)\}$
$\rho\ \mathrm{dc}\,u$		$\rho\,\{\mathrm{dc}\,u + k\,\mathrm{cd}\,u\}$
$k'\rho\ \mathrm{nc}\,u$		$\rho\,\{\mathrm{dc}\,u - k\,\mathrm{cd}\,u\}$
$k'\rho\ \mathrm{sc}\,u$		$\rho\,\{Z_3(u) - Z_2(u)\}$

Transformations of $\rho Z_r(u)$, § 5.

§ 5. The changes in $\rho Z(u)$, $\rho Z_1(u)$, &c. due to the change of q into $-q$ are:

$q,$	$-q,$
$\rho Z(u),$	$\rho Z_3(u),$
$\rho Z_1(u),$	$\rho Z_1(u),$
$\rho Z_2(u),$	$\rho Z_2(u),$
$\rho Z_3(u),$	$\rho Z(u).$

The changes which are due to the change of q into q^2 or $q^{\frac{1}{2}}$ are shown in the following table:

q	q^2	$q^{\frac{1}{2}}$
$\rho Z(u)$	$\frac{1}{2}\{\rho Z_1(u) - \rho \operatorname{sc} u\}$	$\rho Z(2u) + k\rho \operatorname{sn} 2u$
$\rho Z_1(u)$	$\frac{1}{2}\{\rho Z_1(u) + \rho \operatorname{cs} u\}$	$\rho Z_1(2u) + \rho \operatorname{ns} 2u$
$\rho Z_2(u)$	$\frac{1}{2}\{\rho Z_2(u) - k'\rho \operatorname{sc} u\}$	$\rho Z_1(2u) - \rho \operatorname{ns} 2u$
$\rho Z_3(u)$	$\frac{1}{2}\{\rho Z_2(u) + k'\rho \operatorname{sc} u\}$	$\rho Z_1(2u) - k\rho \operatorname{sn} 2u$

The first two quantities in the second column may be expressed in the forms

$$\tfrac{1}{2}\{\rho Z(\tfrac{1}{2}u) + \rho Z_3(\tfrac{1}{2}u)\},\quad \tfrac{1}{2}\{\rho Z_1(\tfrac{1}{2}u) + \rho Z_2(\tfrac{1}{2}u)\}$$

respectively; and the second and third quantities in the third column may be expressed in the forms

$$\rho Z(u) + \rho Z_1(u),\quad \rho Z_2(u) + \rho Z_3(u)$$

respectively.

Transformations of sn u, &c., § 6.

§ 6. It seems desirable to give also the transformations of the functions $\operatorname{sn} u$, $\operatorname{cn} u$, &c.

The changes due to the change of q into $-q$ are:

q,	$-q$,
$\operatorname{sn} u$,	$k' \operatorname{sd} u$,
$\operatorname{cn} u$,	$\operatorname{cd} u$,
$\operatorname{dn} u$,	$\operatorname{nd} u$;
$\operatorname{ns} u$,	$\frac{1}{k'} \operatorname{ds} u$,
$\operatorname{cs} u$,	$\frac{1}{k'} \operatorname{cs} u$,
$\operatorname{ds} u$,	$\frac{1}{k'} \operatorname{ns} u$;
$\operatorname{dc} u$,	$\operatorname{nc} u$,
$\operatorname{nc} u$,	$\operatorname{dc} u$,
$\operatorname{sc} u$,	$k' \operatorname{sc} u$;
$\operatorname{cd} u$,	$\operatorname{cn} u$,
$\operatorname{sd} u$,	$k' \operatorname{sn} u$,
$\operatorname{nd} u$,	$\operatorname{dn} u$;

The changes due to the change of q into q^2 and $q^{\frac{1}{2}}$ are:

q	q^2	$q^{\frac{1}{2}}$
$\operatorname{sn} u$	$(1+k')\dfrac{\operatorname{sn}\frac{1}{2}u\operatorname{cn}\frac{1}{2}u}{\operatorname{dn}\frac{1}{2}u}$	$\dfrac{(1+k)\operatorname{sn} u}{1+k\operatorname{sn}^2 u}$
$\operatorname{cn} u$	$\dfrac{\operatorname{cn}^2\frac{1}{2}u-k'\operatorname{sn}^2\frac{1}{2}u}{\operatorname{dn}\frac{1}{2}u}$	$\dfrac{\operatorname{cn} u\operatorname{dn} u}{1+k\operatorname{sn}^2 u}$
$\operatorname{dn} u$	$\dfrac{\operatorname{cn}^2\frac{1}{2}u+k'\operatorname{sn}^2\frac{1}{2}u}{\operatorname{dn}\frac{1}{2}u}$	$\dfrac{1-k\operatorname{sn}^2 u}{1+k\operatorname{sn}^2 u}$
$\operatorname{ns} u$	$\dfrac{1}{1+k'}\dfrac{\operatorname{dn}\frac{1}{2}u}{\operatorname{sn}\frac{1}{2}u\operatorname{cn}\frac{1}{2}u}$	$\dfrac{1+k\operatorname{sn}^2 u}{(1+k)\operatorname{sn} u}$
$\operatorname{ds} u$	$\dfrac{1}{1+k'}\dfrac{\operatorname{cn}^2\frac{1}{2}u+k'\operatorname{sn}^2\frac{1}{2}u}{\operatorname{sn}\frac{1}{2}u\operatorname{cn}\frac{1}{2}u}$	$\dfrac{1-k\operatorname{sn}^2 u}{(1+k)\operatorname{sn} u}$
$\operatorname{cs} u$	$\dfrac{1}{1+k'}\dfrac{\operatorname{cn}^2\frac{1}{2}u-k'\operatorname{sn}^2\frac{1}{2}u}{\operatorname{sn}\frac{1}{2}u\operatorname{cn}\frac{1}{2}u}$	$\dfrac{\operatorname{cn} u\operatorname{dn} u}{(1+k)\operatorname{sn} u}$
$\operatorname{dc} u$	$\dfrac{\operatorname{cn}^2\frac{1}{2}u+k'\operatorname{sn}^2\frac{1}{2}u}{\operatorname{cn}^2\frac{1}{2}u-k'\operatorname{sn}^2\frac{1}{2}u}$	$\dfrac{1-k\operatorname{sn}^2 u}{\operatorname{cn} u\operatorname{dn} u}$
$\operatorname{nc} u$	$\dfrac{\operatorname{dn}\frac{1}{2}u}{\operatorname{cn}^2\frac{1}{2}u-k'\operatorname{sn}^2\frac{1}{2}u}$	$\dfrac{1+k\operatorname{sn}^2 u}{\operatorname{cn} u\operatorname{dn} u}$
$\operatorname{sc} u$	$(1+k')\dfrac{\operatorname{sn}\frac{1}{2}u\operatorname{cn}\frac{1}{2}u}{\operatorname{cn}^2\frac{1}{2}u-k'\operatorname{sn}^2\frac{1}{2}u}$	$\dfrac{(1+k)\operatorname{sn} u}{\operatorname{cn} u\operatorname{dn} u}$
$\operatorname{cd} u$	$\dfrac{\operatorname{cn}^2\frac{1}{2}u-k'\operatorname{sn}^2\frac{1}{2}u}{\operatorname{cn}^2\frac{1}{2}u+k'\operatorname{sn}^2\frac{1}{2}u}$	$\dfrac{\operatorname{cn} u\operatorname{dn} u}{1-k\operatorname{sn}^2 u}$
$\operatorname{sd} u$	$(1+k')\dfrac{\operatorname{sn}\frac{1}{2}u\operatorname{cn}\frac{1}{2}u}{\operatorname{cn}^2\frac{1}{2}u+k'\operatorname{sn}^2\frac{1}{2}u}$	$\dfrac{(1+k)\operatorname{sn} u}{1-k\operatorname{sn}^2 u}$
$\operatorname{nd} u$	$\dfrac{\operatorname{dn}\frac{1}{2}u}{\operatorname{cn}^2\frac{1}{2}u+k'\operatorname{sn}^2\frac{1}{2}u}$	$\dfrac{1+k\operatorname{sn}^2 u}{1-k\operatorname{sn}^2 u}$

Transformations of $Z(u)$, &c., § 7.

§ 7. It is unnecessary to give the transformations of $Z_s(u)$, as they are deducible at sight from those of $\rho Z_s(u)$ given in § 5. To deduce the transformations of $Z_s(u)$ from those of $\rho Z_s(u)$ it suffices to replace, in the transformed results,

$$\rho \text{ by } \frac{1}{k'} \text{ in the } -q \text{ column,}$$

$$\tfrac{1}{2}\rho \;\;\text{,,}\;\; \frac{1}{1+k'} \;\;\text{,,}\;\; q^2 \;\;\text{,,}$$

$$\rho \;\;\text{,,}\;\; \frac{1}{1+k} \;\;\text{,,}\;\; q^{\frac{1}{2}} \;\;\text{,,}$$

Transformations of ρ, $k\rho$, $k'\rho$, $kk'\rho$, § 8.

§ 8. In connexion with the preceding systems of formulæ it is convenient to give the transformations of ρ, $k\rho$, &c., which are as follows:

q	$-q$	q^2	$q^{\frac{1}{2}}$
ρ	$k'\rho$	$\frac{1}{2}(1+k')\rho$	$(1+k)\rho$
$k\rho$	$ik\rho$	$\frac{1}{2}(1-k')\rho$	$2k^{\frac{1}{2}}\rho$
$k'\rho$	ρ	$k'^{\frac{1}{2}}\rho$	$(1-k)\rho$
$kk'\rho$	$\frac{ik}{k'}\rho$	$k'^{\frac{1}{2}}\frac{1-k'}{1+k'}\rho$	$2k^{\frac{1}{2}}\frac{1-k}{1+k}\rho$

Transformations of $\operatorname{sn} x$, &c., $Z(x)$, &c., § 9.

§ 9. The transformations of the sixteen functions $\operatorname{sn} x$, &c., $Z(x)$, &c., are easily deducible from those of $\operatorname{sn} u$, &c., $Z(u)$, &c., in §§ 6 and 7 by simply multiplying the transformed arguments in the columns headed $-q$, q^2 and $q^{\frac{1}{2}}$ by $\frac{1}{k'}$, $\frac{2}{1+k'}$, and $\frac{1}{1+k}$ respectively.

For example, by the change of q into $-q$, q^2 and $q^{\frac{1}{2}}$, $\operatorname{sn} x$ becomes

$$k' \operatorname{sd}\left(\frac{x}{k'}\right), \quad (1+k')\frac{\operatorname{sn}\dfrac{x}{1+k'}\operatorname{cn}\dfrac{x}{1+k'}}{\operatorname{dn}\dfrac{x}{1+k'}}, \quad \frac{(1+k)\operatorname{sn}\dfrac{x}{1+k}}{1+k\operatorname{sn}^2\dfrac{x}{1+k}}$$

respectively.

The transformations expressed as equations, § 10.

§ 10. By the change of q into $-q$ the modulus k is converted into $\frac{ik}{k'}$ and K into $k'K$. By the changes of q into q^2 and of q into $q^{\frac{1}{2}}$ the modulus k is converted into λ and γ, and K is converted into Λ and Γ, where

$$\lambda = \frac{1-k'}{1+k'}, \qquad \Lambda = \tfrac{1}{2}(1+k')K,$$

$$\gamma = \frac{2k^{\frac{1}{2}}}{1+k}, \qquad \Gamma = (1+k)K.$$

Denoting $\frac{2\Lambda}{\pi}$ and $\frac{2\Gamma}{\pi}$ by ρ_λ and ρ_γ and $\frac{2\Lambda x}{\pi}$ and $\frac{2\Gamma x}{\pi}$ by v and w, the transformations given in the last two sections may be expressed as equations in the form:

$$ik\rho \operatorname{sn}\left(k'u, \frac{ik}{k'}\right) = ikk'\rho \operatorname{sd} u, \quad \&c.,$$

$$\lambda\rho_\lambda \operatorname{sn}(v, \lambda) = \tfrac{1}{2}k^2\rho \frac{\operatorname{sn}\frac{1}{2}u \operatorname{cn}\frac{1}{2}u}{\operatorname{dn}\frac{1}{2}u}, \quad \&c.,$$

$$\gamma\rho_\gamma \operatorname{cn}(w, \gamma) = 2k^{\frac{1}{2}}\rho \frac{(1+k)\operatorname{sn} u}{1+k\operatorname{sn}^2 u}, \quad \&c.,$$

$$\operatorname{sn}\left(x, \frac{ik'}{k}\right) = k' \operatorname{sd}\left(\frac{x}{k'}\right), \quad \&c.,$$

$$\operatorname{sn}(x, \lambda) = (1+k')\frac{\operatorname{sn}\dfrac{x}{1+k'}\operatorname{cn}\dfrac{x}{1+k'}}{\operatorname{dn}\dfrac{x}{1+k'}}, \quad \&c.,$$

$$\operatorname{sn}(x, \gamma) = \frac{(1+k)\operatorname{sn}\dfrac{x}{1+k}}{1+k\operatorname{sn}^2\dfrac{x}{1+k}}, \quad \&c.$$

Expansions of the elliptic functions in powers of x, § 11.

§ 11. The expansions of the twelve elliptic functions in ascending powers of x are as follows:

$$\operatorname{sn} x = x - (1+h)\frac{x^3}{3!} + (1+14h+h^2)\frac{x^5}{5!}$$
$$- (1+135h+135h^2+h^3)\frac{x^7}{7!} + \&c.,$$

$$\operatorname{cn} x = 1 - \frac{x^2}{2!} + (1+4h)\frac{x^4}{4!} - (1+44h+16h^2)\frac{x^6}{6!} + \&c.,$$

$$\operatorname{dn} x = 1 - h\frac{x^2}{2!} + h(h+4)\frac{x^4}{4!} - h(h^2+44h+16)\frac{x^6}{6!} + \&c.;$$

$$\operatorname{ns} x = \frac{1}{x} + \tfrac{1}{6}(1+h)x + \tfrac{1}{60}(7-22h+7h^2)\frac{x^3}{3!}$$
$$+ \tfrac{1}{126}(31-15h-15h^2+31h^3)\frac{x^5}{5!} + \&c.,$$

$$\operatorname{ds} x = \frac{1}{x} - \tfrac{1}{6}(h-h')x + \tfrac{1}{60}(7h^2+22hh'+7h'^2)\frac{x^3}{3!}$$
$$- \tfrac{1}{126}(31h^3+15h^2h'-15hh'^2-31h'^3)\frac{x^5}{5!} + \&c.,$$

$$\operatorname{cs} x = \frac{1}{x} - \tfrac{1}{6}(1+h')x + \tfrac{1}{60}(7-22h'+7h'^2)\frac{x^3}{3!}$$
$$- \tfrac{1}{126}(31-15h'-15h'^2+31h'^3)\frac{x^5}{5!} + \&c.;$$

$$\operatorname{dc} x = 1 + h'\frac{x^2}{2!} + h'(h'+4)\frac{x^4}{4!} + h'(h'^2+44h'+16)\frac{x^6}{6!} + \&c.,$$

$$\operatorname{nc} x = 1 + \frac{x^2}{2!} + (1+4h')\frac{x^4}{4!} + (1+44h'+16h'^2)\frac{x^6}{6!} + \&c.,$$

$$\operatorname{sc} x = x + (1+h')\frac{x^3}{3!} + (1+14h'+h'^2)\frac{x^5}{5!}$$
$$+ (1+135h'+135h'^2+h'^3)\frac{x^7}{7!} + \&c.;$$

$$\operatorname{cd} x = 1 - h'\frac{x^2}{2!} + h'(h'-4h)\frac{x^4}{4!} - h'(h'^2-44hh'+16h^2)\frac{x^6}{6!} + \&c.,$$

$$\operatorname{sd} x = x + (h-h')\frac{x^3}{3!} + (h^2-14hh'+h'^2)\frac{x^5}{5!}$$
$$+ (h^3-135h^2h'+135hh'^2-h'^3)\frac{x^7}{7!} + \&c.,$$

$$\operatorname{nd} x = 1 + h\frac{x^2}{2!} + h(h-4h')\frac{x^4}{4!} + h(h^2-44hh'+16h'^2)\frac{x^6}{6!} + \&c.$$

Expansions of the squares of the elliptic functions in powers of x, § 12.

§ 12. The expansions of sn^2x, sc^2x, sd^2x are:

$$\mathrm{sn}^2x = x^2 - 2^3(1+h)\frac{x^4}{4!} + 2^4(2 \quad + 13h \quad + 2h^2)\frac{x^6}{6!} - \&\text{c.},$$

$$\mathrm{sc}^2x = x^2 + 2^3(1+h')\frac{x^4}{4!} + 2^4(2 \quad + 13h' \quad + 2h'^2)\frac{x^6}{6!} + \&\text{c.},$$

$$\mathrm{sd}^2x = x^2 + 2^3(h-h')\frac{x^4}{4!} + 2^4(2h^2 - 13hh' + 2h'^2)\frac{x}{6!} + \&\text{c.};$$

and that of cs^2x is

$$\mathrm{cs}^2x = \frac{1}{x^2} - \tfrac{1}{3}(1+h') + \tfrac{2}{15}(1-hh')\frac{x^2}{2!}$$
$$+ \tfrac{8}{63}(1+h)(1+h')(h'-h)\frac{x^4}{4!} + \&\text{c.}$$

The expansions of the squares of the other nine functions are deducible at sight from these four series by means of the equations

$$\begin{array}{ll} \mathrm{cn}^2x = 1 - \quad \mathrm{sn}^2x, & \mathrm{dn}^2x = 1 - k^2\,\mathrm{sn}^2x, \\ \mathrm{nc}^2x = 1 + \quad \mathrm{sc}^2x, & \mathrm{dc}^2x = 1 + k'^2\,\mathrm{sc}^2x, \\ \mathrm{nd}^2x = 1 + k^2\mathrm{sd}^2x, & \mathrm{cd}^2x = 1 - k'^2\,\mathrm{sd}^2x, \\ \mathrm{ns}^2x = 1 + \quad \mathrm{cs}^2x, & \mathrm{ds}^2x = k'^2 + \mathrm{cs}^2x. \end{array}$$

The expansions of the squared elliptic functions are deducible at once from those for $\mathrm{iz}_r x$, $\mathrm{gz}_r x$, $\mathrm{ez}_r x$, or *vice versâ*, by the means of the formulæ $\mathrm{iz}'x = -k^2\,\mathrm{sn}^2x$, &c., or of the converse formulæ (vol. XV., p. 112).

Expansions of the Zeta Functions in ascending powers of x, §§ 13–16.

§ 13. The first four terms of the expansions of the functions $\mathrm{gz}_r x$ in ascending powers of x are:*

$$\mathrm{gz}\,x = h\left\{x - 2\frac{x^3}{3!} + 2^3(1+h)\frac{x^5}{5!} - 2^4(2+13h+2h^2)\frac{x^7}{7!} + \&\text{c.}\right\},$$

$$\mathrm{gz}_1x = \frac{1}{x} + \tfrac{1}{3}(h-h')x - \tfrac{2}{15}(1-hh')\frac{x^3}{3!}$$
$$+ \tfrac{8}{63}(1+h)(1+h')(h-h')\frac{x^5}{5!} - \&\text{c.},$$

---

* *Messenger*, vol. XV. p. 144.

$$\mathrm{gz}_2 x = -h'\left\{x + 2\frac{x^3}{3!} + 2^3(1+h')\frac{x^5}{5!} + 2^4(2+13h'+2h'^2)\frac{x^7}{7!} + \&c.\right\},$$

$$\mathrm{gz}_3 x = hh'\left\{2\frac{x^3}{3!} + 2^3(h-h')\frac{x^5}{5!} + 2^4(2h^2-13hh'+2h'^2)\frac{x^7}{7!} + \&c.\right\}.$$

§ 14. The other Zeta functions $\mathrm{iz}_s x$, &c., differ from $\mathrm{gz}_s x$ only by multiples of x, so that with the sole exception of the term involving x the expansions of these functions are the same as those of $\mathrm{gz}_s x$. The accompanying table gives the value of the term involving x for each of the twelve functions $\mathrm{iz}_s x$, $\mathrm{gz}_s x$, $\mathrm{ez}_s x$.

s	$\mathrm{iz}_s x$	$\mathrm{gz}_s x$	$\mathrm{ez}_s x$
0	0	hx	x
1	$-\frac{1}{3}(1+h)x$	$\frac{1}{3}(h-h')x$	$\frac{1}{3}(1+h')x$
2	$-x$	$-h'x$	0
3	$-hx$	0	$h'x$

The corresponding terms in the expansions of $Z_s(x)$ and $\zeta_s(x)$ are shown in the next table.

s	$Z_s(x)$	$\zeta_s(x)$
0	$-\frac{I}{K}x$	$\frac{I'}{K'}x$
1	$-\frac{H}{K}x$	$\frac{H'}{K'}x$
2	$-\frac{E}{K}x$	$\frac{E'}{K'}x$
3	$-\frac{G}{K}x$	$\frac{G'}{K'}x$

The functions $\zeta_\epsilon(x)$ are the functions so designated in vol. xv. pp. 92–102. The quantities H and H' denote $\frac{1}{3}(I+G+E)$ and $\frac{1}{3}(I'+G'+E')$ respectively.

§ 15. It will be noticed that the series for gz$x$, gz$_1x$, gz$_2x$ contain h, h', hh' respectively as factors. The only other series which contain a factor common to all the terms are those for iz$x$ and ez$_2x$, neither of which contains a term in x.

The three series which contain no term in x are:

$$\text{iz}\,x = - \quad h\left\{2\frac{x^3}{3!} - 2^3(1+h)\frac{x^5}{5!} + 2^4(2 \quad +13h \quad +2h^2)\frac{x^7}{7!} - \&\text{c.}\right\},$$

$$\text{ez}_2x = - \quad h'\left\{2\frac{x^3}{3!} + 2^3(1+h')\frac{x^5}{5!} + 2^4(2 \quad +13h' \quad +2h'^2)\frac{x^7}{7!} + \&\text{c.}\right\},$$

$$\text{gz}_2x = \quad hh'\left\{2\frac{x^3}{3!} + 2^3(h-h')\frac{x^5}{5!} + 2^4(2h^2 - 13hh' + 2h'^2)\frac{x^7}{7!} + \&\text{c.}\right\}.$$

§ 16. The values of $Z_\epsilon(x)$, when x is small, are often required in verifying formulæ and for other purposes. I therefore add for the sake of reference the expansions of $Z_\epsilon(x)$ as far as the terms involving x^3:

$$Z(x) = \quad -\frac{I}{K}x - \tfrac{1}{3}hx^3,$$

$$Z_1(x) = \frac{1}{x} - \frac{H}{K}x - \tfrac{1}{45}(1-hh')\,x^3,$$

$$Z_2(x) = \quad -\frac{E}{K}x - \tfrac{1}{3}h'x^3,$$

$$Z_3(x) = \quad -\frac{G}{K}x + \tfrac{1}{3}hh'x^3.$$

Increase of the argument by K, iK', $K+iK'$, § 17.

§ 17. In working with the twelve elliptic functions it is convenient to have for reference in a tabular form the complete system of changes which are produced in the

functions by the increase of the argument by K, iK' and $K+iK'$.

These changes are shown in the following table:

x	$x+K$	$x+iK'$	$x+K+iK'$
$\operatorname{sn} x$	$\operatorname{cd} x$	$\frac{1}{k} \operatorname{ns} x$	$\frac{1}{k} \operatorname{dc} x$
$\operatorname{cn} x$	$-k' \operatorname{sd} x$	$-\frac{i}{k} \operatorname{ds} x$	$-\frac{ik'}{k} \operatorname{nc} x$
$\operatorname{dn} x$	$k' \operatorname{nd} x$	$-i \operatorname{cs} x$	$ik' \operatorname{sc} x$
$\operatorname{ns} x$	$\operatorname{dc} x$	$k \operatorname{sn} x$	$k \operatorname{cd} x$
$\operatorname{ds} x$	$k' \operatorname{nc} x$	$-ik \operatorname{cn} x$	$ikk' \operatorname{sd} x$
$\operatorname{cs} x$	$-k' \operatorname{sc} x$	$-i \operatorname{dn} x$	$-ik' \operatorname{nd} x$
$\operatorname{dc} x$	$-\operatorname{ns} x$	$k \operatorname{cd} x$	$-k \operatorname{sn} x$
$\operatorname{nc} x$	$-\frac{1}{k'} \operatorname{ds} x$	$ik \operatorname{sd} x$	$\frac{ik}{k'} \operatorname{cn} x$
$\operatorname{sc} x$	$-\frac{1}{k'} \operatorname{cs} x$	$i \operatorname{nd} x$	$\frac{i}{k'} \operatorname{dn} x$
$\operatorname{cd} x$	$-\operatorname{sn} x$	$\frac{1}{k} \operatorname{dc} x$	$-\frac{1}{k} \operatorname{ns} x$
$\operatorname{sd} x$	$\frac{1}{k'} \operatorname{cn} x$	$\frac{i}{k} \operatorname{nc} x$	$-\frac{i}{kk'} \operatorname{ds} x$
$\operatorname{nd} x$	$\frac{1}{k'} \operatorname{dn} x$	$i \operatorname{sc} x$	$-\frac{i}{k'} \operatorname{cs} x$

The next table, which is deducible at once from the last, gives the corresponding changes of $k \operatorname{sn} x$, $k \operatorname{cn} x$, &c.

These twelve functions form a group complete in itself, viz. each function is transformed into another member of the group multiplied by ± 1 or $\pm i$.

x	$x+K$	$x+iK'$	$x+K+iK'$
$k\,\mathrm{sn}\,x$	$k\,\mathrm{cd}\,x$	$\mathrm{ns}\,x$	$\mathrm{dc}\,x$
$k\,\mathrm{cn}\,x$	$-kk'\,\mathrm{sd}\,x$	$-i\,\mathrm{ds}\,x$	$-ik'\,\mathrm{nc}\,x$
$\mathrm{dn}\,x$	$k'\,\mathrm{nd}\,x$	$-i\,\mathrm{cs}\,x$	$ik'\,\mathrm{sc}\,x$
$\mathrm{ns}\,x$	$\mathrm{dc}\,x$	$k\,\mathrm{sn}\,x$	$k\,\mathrm{cd}\,x$
$\mathrm{ds}\,x$	$k'\,\mathrm{nc}\,x$	$-ik\,\mathrm{cn}\,x$	$ikk'\,\mathrm{sd}\,x$
$\mathrm{cs}\,x$	$-k'\,\mathrm{sc}\,x$	$-i\,\mathrm{dn}\,x$	$-ik'\,\mathrm{nd}\,x$
$\mathrm{dc}\,x$	$-\mathrm{ns}\,x$	$k\,\mathrm{cd}\,x$	$-k\,\mathrm{sn}\,x$
$k'\,\mathrm{nc}\,x$	$-\mathrm{ds}\,x$	$ikk'\,\mathrm{sd}\,x$	$ik\,\mathrm{cn}\,x$
$k'\,\mathrm{sc}\,x$	$-\mathrm{cs}\,x$	$ik'\,\mathrm{nd}\,x$	$i\,\mathrm{dn}\,x$
$k\,\mathrm{cd}\,x$	$-k\,\mathrm{sn}\,x$	$\mathrm{dc}\,x$	$-\mathrm{ns}\,x$
$kk'\,\mathrm{sd}\,x$	$k\,\mathrm{cn}\,x$	$ik'\,\mathrm{nc}\,x$	$-i\,\mathrm{ds}\,x$
$k'\,\mathrm{nd}\,x$	$\mathrm{dn}\,x$	$ik'\,\mathrm{sc}\,x$	$-i\,\mathrm{cs}\,x$

Values of the elliptic functions when the argument is $0, K, iK', K+iK'$, § 18.

§ 18. The following table gives the values of the twelve elliptic functions for the arguments $0, K, iK', K+iK'$.

The letter a denotes zero and A infinity, but the following more precise significations may be attached to these letters, viz. the arguments $0, K, iK', K+iK'$ may be regarded as denoting $a, K+a, iK'+a, K+iK'+a$, where a is infinitesimal, and A denotes $\frac{1}{a}$.

Thus the column headed 0 shows that, a being infinitesimal,

$$\operatorname{sn} a = a, \quad \operatorname{ns} a = \frac{1}{a}, \quad \operatorname{ds} a = \frac{1}{a}, \quad \&c.$$

The column headed K shows that

$$\operatorname{cn}(K+a) = -k'a, \quad \operatorname{dc}(K+a) = -\frac{1}{a}, \quad \&c.,$$

and so on.

x	0	K	iK'	$K+iK'$
$\operatorname{sn} x$	a	1	$\frac{1}{k}A$	$\frac{1}{k}$
$\operatorname{cn} x$	1	$-k'a$	$-\frac{i}{k}A$	$-\frac{ik'}{k}$
$\operatorname{dn} x$	1	k'	$-iA$	$ik'a$
$\operatorname{ns} x$	A	1	ka	k
$\operatorname{ds} x$	A	k'	$-ik$	$ikk'a$
$\operatorname{cs} x$	A	$-k'a$	$-i$	$-ik'$
$\operatorname{dc} x$	1	$-A$	k	$-ka$
$\operatorname{nc} x$	1	$-\frac{1}{k'}A$	$ik'a$	$\frac{ik}{k'}$
$\operatorname{sc} x$	a	$-\frac{1}{k'}A$	i	$\frac{i}{k'}$
$\operatorname{cd} x$	1	$-a$	$\frac{1}{k}$	$-\frac{1}{k}A$
$\operatorname{sd} x$	a	$\frac{1}{k'}$	$\frac{i}{k}$	$-\frac{i}{kk'}A$
$\operatorname{nd} x$	1	$\frac{1}{k'}$	ia	$-\frac{i}{k'}A$

The table therefore gives not only the values of the functions for the arguments, 0, K, &c., but also the ratio to one another in which they vanish and become infinite.

Increase of the argument by $2K$, $2iK'$, $2K+2iK'$, § 19.

§ 19. The next table gives the signs by which the twelve functions are affected when the argument is increased by $2K$, $2iK'$ and $2K+2iK'$. For example,

$$\operatorname{sn}(x+2K) = -\operatorname{sn} x,$$
$$\operatorname{sn}(x+2iK') = +\operatorname{sn} x,$$
$$\operatorname{sn}(x+2K+2iK') = -\operatorname{sn} x,$$

and so on.

x	$x+2K$	$x+2iK'$	$x+2K+2iK'$	0
sn x	−	+	−	a
cn x	−	−	+	1
dn x	+	−	−	1
ns x	−	+	−	A
ds x	−	−	+	A
cs x	+	−	−	A
dc x	−	+	−	1
nc x	−	−	+	1
sc x	+	−	−	a
cd x	−	+	−	1
sd x	−	−	+	a
nd x	+	−	−	1

In the last column the values of the functions for $x=a$, a being infinitesimal, are added. By the aid of the three previous columns the values of the functions $2K+a$, $2iK'+a$, $2K+2iK'+a$ for a infinitesimal may be written down at sight by affixing the proper sign; for example

$$\operatorname{sn}(2K+a)=-a,\quad \operatorname{sn}(2iK'+a)=+a,\quad \operatorname{sn}(2K+2iK'+a)=-a,$$
$$\operatorname{cn}(2K+a)=-1,\quad \operatorname{cn}(2iK'+a)=-1,\quad \operatorname{cn}(2K+2iK'+a)=+1;$$
&c., &c.

SYSTEM OF EQUATIONS FOR THREE CIRCLES WHICH CUT EACH OTHER AT GIVEN ANGLES.

By Prof. *Cayley*.

CONSIDER a triangle ABC, angles A, B, C $(A+B+C=\pi)$: to fix the absolute magnitude, assume that the radius of the circumscribed circle is $=1$, the lengths of the sides are therefore $=2\sin A$, $2\sin B$, $2\sin C$ respectively. On the three sides as bases, outside of each, describe isosceles triangles aBC, bCA, cAB, the base angles whereof are $=\alpha$, β, γ respectively. If we draw a circle touching aB, aC at the points B, C respectively; a circle touching bC, bA at the points C, A respectively; and a circle touching cA, cB at the points A, B respectively; then these circles form a curvilinear triangle ABC, the angles whereof are $A+\beta+\gamma$, $B+\gamma+\alpha$, $C+\alpha+\beta$ respectively. Taking as origin the centre of the circumscribed circle, and through this point, for axis of x, an arbitrary line, its position determined by the angle θ, I write for convenience

$$\begin{array}{lll} F=\theta+2B, & F'=\theta-A, & A'=A+\beta+\gamma, \\ G=\theta+2B+2C, & G'=\theta+B, & B'=B+\gamma+\alpha, \\ H=\theta, & H'=\theta+2B+C, & C'=C+\alpha+\beta; \end{array}$$

then the coordinates of the angular points A, B, C are $(\cos F, \sin F)$, $(\cos G, \sin G)$ $(\cos H, \sin H)$ respectively; and the equations of the three circles are

$$\left(x+\frac{\sin(A-\alpha)}{\sin\alpha}\cos F'\right)^2+\left(y+\frac{\sin(A-\alpha)}{\sin\alpha}\sin F'\right)^2=\frac{\sin^2 A}{\sin^2\alpha},$$
$$\left(x+\frac{\sin(B-\beta)}{\sin\beta}\cos G'\right)^2+\left(y+\frac{\sin(B-\beta)}{\sin\beta}\sin G'\right)^2=\frac{\sin^2 B}{\sin^2\beta},$$
$$\left(x+\frac{\sin(C-\gamma)}{\sin\gamma}\cos H'\right)^2+\left(y+\frac{\sin(C-\gamma)}{\sin\gamma}\sin H'\right)^2=\frac{\sin^2 C}{\sin^2\gamma},$$

respectively.

In verification observe that we have

$$\begin{array}{llll} G-H=2\pi-2A, & G'-H'=-\pi+A, & G-F'=2\pi-A, & H-F'=A, \\ H-F=\ \ -2B, & H'-F'=\ \ \pi+B, & H-G'=\ \ -B, & F-G'=B, \\ F-G=\ \ -2C, & F'-G'=-\pi+C, & F-H'=\ \ -C, & G-H'=C, \end{array}$$

hence

$$(\cos G-\cos H)^2+(\sin G-\sin H)^2=2-2\cos(G-H),$$
$$=2(1-\cos 2A),\ =4\sin^2 A;$$

and we thus see that the sides are $=2\sin A$, $2\sin B$, $2\sin C$ respectively.

The first circle should pass through the points $(\cos G, \sin G)$, $(\cos H, \sin H)$ we ought therefore to have for the first of these points

$$1+2\frac{\sin(A-\alpha)}{\sin\alpha}\cos(G-F)+\frac{\sin^2(A-\alpha)}{\sin^2\alpha}=\frac{\sin^2 A}{\sin^2\alpha},$$

that is,

$$1+2\frac{\sin(A-\alpha)}{\sin\alpha}\cos A+\frac{\sin^2(A-\alpha)}{\sin^2\alpha}=\frac{\sin^2 A}{\sin^2\alpha},$$

and for the second of the points the same equation. Write for a moment

$$X=\frac{\sin A}{\sin\alpha},\ \text{then}\ \frac{\sin(A-\alpha)}{\sin\alpha}=X\cos\alpha-\cos A,$$

and the equation is

$$1+2(X\cos\alpha-\cos A)\cos A+(X\cos\alpha-\cos A)^2=X^2,$$

that is, $$1-\cos^2 A=X^2\sin^2\alpha,$$

which is right.

The second and third circles should intersect at the angle A', that is we ought to have

$$\left(\frac{\sin(B-\beta)}{\sin\beta}\cos G'-\frac{\sin(C-\gamma)}{\sin\gamma}\cos H'\right)^2$$
$$+\left(\frac{\sin(B-\beta)}{\sin\beta}\sin G'-\frac{\sin(C-\gamma)}{\sin\gamma}\sin H'\right)^2$$
$$=\frac{\sin^2 B}{\sin^2\beta}+\frac{\sin^2 C}{\sin^2\gamma}+2\frac{\sin B\sin C}{\sin\beta\sin\gamma}\cos A',$$

or reducing and for $\cos(G'-H')$ substituting its value, $=-\cos A$, the equation is

$$\frac{\sin^2(B-\beta)}{\sin^2\beta}+\frac{\sin^2(C-\gamma)}{\sin^2\gamma}+2\frac{\sin(B-\beta)\sin(C-\gamma)}{\sin\beta\sin\gamma}\cos A$$
$$=\frac{\sin^2 B}{\sin^2\beta}+\frac{\sin^2 C}{\sin^2\gamma}+2\frac{\sin B\sin C}{\sin\beta\sin\gamma}\cos A'.$$

Writing here

$$\frac{\sin B}{\sin \beta} = Y, \ \frac{\sin B}{\sin \gamma} = Z,$$

the equation is

$$(Y\cos\beta - \cos B)^2 + (Z\cos\gamma - \cos C)^2$$
$$+ 2\,(Y\cos\beta - \cos B)\,(Z\cos\gamma - \cos C)\cos A$$
$$= Y^2 + Z^2 + 2\,YZ\cos A',$$

viz. this is,

$$Y^2\cos^2\beta + Z^2\cos^2\gamma + 2\,YZ\cos\beta\cos\gamma\cos A$$
$$- 2\,Y\cos\beta\,(\cos B + \cos C\cos A) - 2Z\cos\gamma\,(\cos C + \cos A\cos B)$$
$$+ \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C$$
$$= Y^2 + Z^2 + 2\,YZ\cos A'.$$

Reducing by the relation $A + B + C = \pi$, this becomes

$$-2\,Y\cos\beta\sin A\sin C - 2Z\cos\gamma\sin A\sin B + 1 - \cos^2 A$$
$$= Y^2\sin^2\beta + Z^2\sin^2\gamma + 2\,YZ\,(\cos A' - \cos\beta\cos\gamma\cos A).$$

Here $A' = A + \beta + \gamma$, and thence

$$\cos A' = \cos A\,(\cos\beta\cos\gamma - \sin\beta\sin\gamma)$$
$$- \sin A\,(\sin\gamma\cos\beta + \sin\beta\cos\gamma),$$

and hence the right hand is

$$= Y^2\sin^2\beta + Z^2\sin^2\gamma$$
$$-2\,YZ(\cos A\sin\beta\sin\gamma + \sin A\sin\gamma\cos\beta + \sin A\sin\beta\cos\gamma)$$

or reducing by

$$Y\sin\beta = \sin B,\ \ Z\sin\gamma = \sin C,$$

this is

$$= \sin^2 B + \sin^2 C - 2\sin B\sin C\cos A$$
$$-2\,Y\cos\beta\sin A\sin C - 2Z\cos\gamma\sin A\sin B,$$

and the terms in Y, Z are equal to the like terms on the left-hand; the whole equation thus becomes

$$-1 + \cos^2 A + \sin^2 B + \sin^2 C - 2\cos A\sin B\sin C = 0,$$

where the last term is

$$= 2\cos A\,\{\cos(B + C) - \cos B\cos C\},$$
$$= -2\cos^2 A - 2\cos A\cos B\cos C,$$
$$= -2\cos^2 A + (\cos^2 A + \cos^2 B + \cos^2 C - 1),$$
$$= -\cos^2 A + \cos^2 B + \cos^2 C - 1\,;$$

the equation is thus

$$-1+\cos^2 A+\sin^2 B+\sin^2 C-\cos^2 A+\cos^2 B+\cos^2 C-1=0,$$

or finally it is $-1+1+1-1=0$, which is an identity. The formulæ for the intersection of the third and first circles, and for that of the first and second circles, are of course precisely similar to the above formula for the intersection of the second and third circles; and the verifications are thus completed.

Cambridge, *April* 7, 1887.

NOTE ON THE LEGENDRIAN COEFFICIENTS OF THE SECOND KIND.

By Prof. *Cayley*.

As regards the integration of the equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

(n a positive integer), it seems to me that sufficient prominence is not given to the solution

$$y=\tfrac{1}{2}P_n\log\frac{x+1}{x-1}-Z_n\,(=Q_n),$$

where P_n is the Legendrian integral of the first kind, a rational and integral function of x of the degree n, and Z_n is a rational and integral function of the degree $n-1$; viz. we have here a solution containing no transcendental function other than the logarithm, and which should thus be adopted as a second particular integral in preference to the form $y=Q_n$ in which we have the infinite series Q_n which is an unknown transcendental function.

Moreover, the expression usually given for Z_n, viz.,

$$Z_n=\frac{2n-1}{1.n}P_{n-1}+\frac{2n-5}{3(n-1)}P_{n-3}+\frac{2n-9}{5(n-2)}P_{n-5}\ldots$$

(to term in P_1 or P_0),

is a very simple and elegant one; but the more natural definition (and that by which Z_n is most readily calculated) is Z_n is the integral part of $\frac{1}{2}P_n\log\frac{x+1}{x-1}$, when the logarithm

is expanded in descending powers of x, viz. it is the integral part of

$$P_n\left(\frac{1}{x}+\frac{1}{3}\frac{1}{x^3}+\frac{1}{5}\frac{1}{x^5}+\ldots\right)$$

(whence also Q_n is the portion containing negative powers only of this same series).

The expressions for P_0, P_1,...P_{10} are given in Ferrers' "*Elementary Treatise on Spherical Harmonics, &c.*" London 1877, pp. 23–25. Reproducing these, and joining to them the values of Z_0, Z_1...Z_{10} we have as follows: read $P_2=\frac{3}{2}x^2-\frac{1}{2}$, and so in other cases.

$$\begin{array}{lllllll}
P_0 = & 1\ , & & & & & \\
P_1 = x & 1\ , & & & & & \\
P_2 = (x^2,\ 1) & \frac{3}{2} & -\ \frac{1}{2}\ , & & & & \\
P_3 = (x^3,\ x) & \frac{5}{2} & -\ \frac{3}{2}\ , & & & & \\
P_4 = (x^4\ldots 1) & \frac{35}{8} & -\ \frac{15}{4} & +\ \frac{3}{8}\ , & & & \\
P_5 = (x^5\ldots x) & \frac{63}{8} & -\ \frac{35}{4} & +\ \frac{15}{8}\ , & & & \\
P_6 = (x^6\ldots 1) & \frac{231}{16} & -\ \frac{315}{16} & +\ \frac{105}{16} & -\ \frac{5}{16}\ , & & \\
P_7 = (x^7\ldots x) & \frac{429}{16} & -\ \frac{693}{16} & +\ \frac{315}{16} & -\ \frac{35}{16}\ , & & \\
P_8 = (x^8\ldots 1) & \frac{6435}{128} & -\ \frac{3003}{32} & +\ \frac{3465}{64} & -\ \frac{315}{32} & +\ \frac{35}{128}\ , & \\
P_9 = (x^9\ldots x) & \frac{12155}{128} & -\ \frac{6435}{32} & +\ \frac{9009}{64} & -\ \frac{1155}{32} & +\ \frac{315}{128}\ , & \\
P_{10} = (x^{10}\ldots 1) & \frac{46189}{256} & -\ \frac{109395}{256} & +\ \frac{45045}{128} & -\ \frac{15015}{128} & +\ \frac{3465}{256} & -\ \frac{63}{256}. \\
Z_0 = & 0\ , & & & & & \\
Z_1 = & 1\ , & & & & & \\
Z_2 = x & \frac{3}{2}\ , & & & & & \\
Z_3 = (x^2,\ 1) & \frac{5}{2} & -\ \frac{2}{3}\ , & & & & \\
Z_4 = (x^3,\ x) & \frac{35}{8} & -\ \frac{55}{24}\ , & & & & \\
Z_5 = (x^4\ldots 1) & \frac{63}{8} & -\ \frac{49}{8} & +\ \frac{8}{15}\ , & & & \\
Z_6 = (x^5\ldots x) & \frac{231}{16} & -\ \frac{119}{8} & +\ \frac{231}{80}\ , & & & \\
Z_7 = (x^6\ldots 1) & \frac{429}{16} & -\ \frac{275}{8} & +\ \frac{849}{80} & -\ \frac{16}{35}\ , & & \\
Z_8 = (x^7\ldots x) & \frac{6435}{128} & -\ \frac{9867}{128} & +\ \frac{4213}{128} & -\ \frac{11659}{4480}\ , & & \\
Z_9 = (x^8\ldots 1) & \frac{12155}{128} & -\ \frac{65065}{384} & +\ \frac{11869}{128} & -\ \frac{14179}{896} & +\ \frac{128}{315}\ , & \\
Z_{10} = (x^9\ldots x) & \frac{46189}{256} & -\ \frac{281996}{768} & +\ \frac{157157}{640} & -\ \frac{26741}{448} & +\ \frac{61567}{16128}. &
\end{array}$$

I notice that the numerical values of P_1, P_2, ..., P_7, for $x=0{\cdot}00$, $0{\cdot}01$, ..., $1{\cdot}00$ are given (*Report of the British Association for* 1879, "Report on Mathematical Tables"); as the functions contain only powers of 2 in their denominators, the decimal values terminate, and the complete values are given. The functions Z have not been tabulated, the denominators contain other prime factors, and the decimal values would not terminate.

Cambridge, *March* 29, 1887.

THE TRANSFORMATION OF MULTIPLE SURFACE INTEGRALS INTO MULTIPLE LINE INTEGRALS.

By *J. Larmor.*

An integral extended throughout a volume can in various ways be expressed as a surface integral over its boundary. Many elegant theorems of this kind have been given by Gauss.*

I. But in order that the integral over a surface, of a vector function, meaning thereby the integral of its normal component over the surface, may be expressible by a line integral over its contour, the function must satisfy a certain condition.

In fact the integrals over any two surfaces abutting on the same contour would then be equal, and the two together would form a closed surface, such that the integral taken in the same sense over the whole of it would be equal to zero. Now if R denote the vector, X, Y, Z its components parallel to the axes, and $R\cos\epsilon$ its normal component,

$$\iint R\cos\epsilon\, dS = \iiint\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)d\text{ vol.} \quad\text{......(1)}$$

Therefore if this condition of zero integral is to hold for all closed surfaces, we must have identically, throughout the space considered,

$$\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}=0, \quad\text{..................(2)}$$

as the condition required.

---

* *Theoria Attractionis...*, *Comm. Soc. Götting.*, II, 1813, or *Werke.* Band V.

The truth of the formula (1) requires that the vector should not become discontinuous or its differential coefficients infinite anywhere in the space in question; for if that were not provided for, the integration of its right-hand side might introduce other terms: cf. Maxwell's *Electricity*, Ch. I.

The proposition must therefore be applied in its simple form, only when the region in question does not contain places where the vector is discontinuous or its differential coefficients infinite.

If X, Y, Z are the components of a flux R, the condition (2) is the well-known "Equation of Continuity," which secures that the flux is that of an incompressible substance. Thus in continuous motion of incompressible fluids the flux through any ideal aperture is expressible as a line integral round its contour; the reason for which is obvious.

To determine the form of the integral relation in question, we may first take the case of a small plane surface.

Then
$$\int(\alpha\,dx+\beta\,dy)=\iint dx\,dy\left(\frac{d\beta}{dx}-\frac{d\alpha}{dy}\right)\ldots\ldots\ldots(3)$$

by immediate integration, the rule of signs being that the line integral proceeds round the contour in the direction from x to y in the first quadrant.

In the same way, for areas in the planes of yz and zx, we have

$$\int(\beta\,dy+\gamma\,dz)=\iint dy\,dz\left(\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right),\ldots\ldots\ldots(4)$$

$$\int(\gamma\,dz+\alpha\,dx)=\iint dz\,dx\left(\frac{d\alpha}{dz}-\frac{d\gamma}{dx}\right).\ \ldots\ldots\ldots(5)$$

By what precedes, expressions to be integrated on the right-hand are to be taken as the components normal to the coordinate planes of the vector function R; and we remark that they satisfy (2).

We are entitled therefore to assert for any small plane contour, that

$$\int(\alpha\,dx+\beta\,dy+\gamma\,dz)$$
$$=\iint dS.R\cos\epsilon\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (6)$$

where the components of R are

$$X=\frac{d\gamma}{dy}-\frac{d\beta}{dz},\quad Y=\frac{d\alpha}{dz}-\frac{d\gamma}{dx},\quad Z=\frac{d\beta}{dx}-\frac{d\alpha}{dy}.\ \ldots(7)$$

And by adding the results for the series of infinitesimal plane circuits into which any finite circuit may be divided, we see that the theorem, due originally to Stokes, holds for a contour of any form.

The rule of signs now is that the direction of integration round the contour corresponds to that of a right-handed screw along the direction of R. For this rule is in agreement with the constituent formulae (3), (4), (5), when the system of axes form a right-handed system, as it always should in such directional investigations, viz. when the directions of rotation in the positive quadrant from y to z, z to x, and x to y correspond to right-handed screws along the positive directions of the other axes.

When there is discontinuity in X, Y, Z such that for any region included in the space between two surfaces (A) and (B) abutting on the same contour,

$$\iiint\left(\frac{dX}{dx}+\frac{dY}{dy}+\frac{dZ}{dz}\right)d\,\text{vol.} \;\ldots\ldots\ldots\ldots \quad (8)$$

is not zero, the surface integrals over the two surfaces are no longer equal, but differ by the value of the expression (8).

The explanation of this discontinuity is most clearly seen from the representation as a flux. The value of (8) extended over any region then denotes an emission of fluid in that region at a rate per second given by that expression. If we consider the simplest case of fluid welling out at a single point, then as the surface (A) is gradually altered into (B), when it passes over that point the direction of the velocity due to the source there situated is changed with respect to the surface, and a finite alteration is thereby produced at that stage in the value of the surface integral.

Thus if
$$R=\frac{1}{r^2},$$
so that
$$X=\frac{x}{r^3},\quad Y=\frac{y}{r^3},\quad Z=\frac{z}{r^3},$$
the condition (2) is satisfied, and it is not difficult to verify that

$$\iint R\cos\varepsilon\,dS$$
$$=\int\frac{x\,(ydz-zdy)}{r\,(y^2+z^2)}-A, \;\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

where A is equal to zero or 4π according to the side of the origin on which the surface bounded by the given contour lies.

We may obtain other forms for the theorem by adding to the right-hand side of (9) the line integral of any exact differential, which will add nothing when taken round the circuit.

This formula (9) expresses as a line integral the flux due to a single source of fluid at the origin of coordinates, or the induction due to a single attracting particle situated there; and from it any more general case might be deduced by summation. But development in this direction simply leads to the well-known theory of the vector potential in Electro-dynamics.

II. There is another class of integrals related to Mathematical Physics in which the integrals are extended over two contours. For instance, a uniformly luminous open surface emits a quantity of radiation through a given aperture which depends only on the contours of the surface and aperture, care being taken that all parts of one contour are visible from all parts of the other. Again, the mutual energy of two closed electric currents may be expressed either as an integral extended over their circuits, or as a surface integral derived from the equivalent magnetic shells.

We propose now to investigate the general forms of such relations.

If a line integral round a contour is to be expressible as a surface integral over a sheet bounded by the contour, by means of (6), it must involve the elements of the contour linearly. Therefore the most general type of double line integral in question must involve both contours linearly. The function to be integrated can only involve the distance between two elements of the contours and the mutual inclinations of the distance and these elements. If r denote the distance of the elements ds, ds', and ϑ, ϑ' the angles it makes with these elements, and ε the angle between the directions of the elements, the most general forms therefore involve only

$$\iint ds\,ds' f(r) \cos\varepsilon, \quad\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

and $$\iint ds\,ds' \phi(r) \cos\vartheta \cos\vartheta'. \quad\ldots\ldots\ldots\ldots\ldots (11)$$

Of these the latter clearly vanishes when either circuit is complete.

The former

$$= \int ds' \int f(r)\,(l'dx + m'dy + n'dz),$$

where l', m', n' are the direction cosines of ds';

$$= \int ds' \iint dS\,(X\lambda + Y\mu + Z\nu), \text{ by (6)},$$

where λ, μ, ν are the direction cosines of the normal to dS, and

$$X = f'(r)\frac{yn' - zm'}{r},$$

$$Y = f'(r)\frac{zl' - xn'}{r},$$

$$Z = f'(r)\frac{xm' - yl'}{r},$$

x, y, z being the components of r, the origin being taken temporarily at the position of ds'.

Thus changing the order of integration, and transferring the origin to the position of dS, so that we write $-x'$, $-y'$, $-z'$ for x, y, z, we have

$$\iint dS \int f'(r)\left(\frac{y'\nu - z'\mu}{r}dx' + \frac{z'\lambda - x'\nu}{r}dy' + \frac{x'\mu - y'\lambda}{r}dz'\right)$$

$$= \iint dS \int(\alpha' dx' + \beta' dy' + \gamma' dz'), \text{ say},$$

$$= \iint dS \iint dS'(X'\lambda' + Y'\mu' + z'\nu'),$$

where $$X' = \frac{d\gamma'}{dy'} - \frac{d\beta'}{dz'}, \text{ by (7)},$$

$$= \frac{d}{dr}\left\{\frac{1}{r}f'(r)\right\}\left[\frac{y'}{r}(x'\mu - y'\lambda) - \frac{z'}{r}(z'\lambda - x'\nu) + \ldots + \ldots\right]$$

$$+ \frac{1}{r}f'(r)[-2\lambda - \ldots - \ldots];$$

so that $$X'\lambda' + Y'\mu' + Z'\nu'$$

$$= \frac{1}{r}\frac{d}{dr}\left\{\frac{1}{r}f'(r)\right\}[-(y'^2 + z'^2)\lambda\lambda' + x'\lambda(y'\mu' + z'\nu') + \ldots + \ldots]$$

$$- \frac{2}{r}f'(r)[\lambda\lambda' + \mu\mu' + \nu\nu']$$

$$= r\frac{d}{dr}\left\{\frac{1}{r}f'(r)\right\}[-\cos\eta + \cos\theta\cos\theta'] - \frac{2}{r}f'(r)\cos\eta$$

$$= -\left\{f''(r) + \frac{1}{r}f'(r)\right\}\cos\eta + \left\{f''(r) - \frac{1}{r}f'(r)\right\}\cos\theta\cos\theta',$$

$$= -\frac{1}{r}\frac{d}{dr}\{rf'(r)\}\cos\eta + r\frac{d}{dr}\left\{\frac{1}{r}f'(r)\right\}\cos\theta\cos\theta',$$

where η is the angle between the normals to dS, dS' each drawn towards the positive side of the surface, and θ, θ' are the angles between these normals and r, whose direction is the same in both cases.

Therefore, finally,

$$\iint dS \iint dS' \left[r\frac{d}{dr}\left\{\frac{1}{r}f'(r)\right\}\cos\theta\cos\theta' - \frac{1}{r}\frac{d}{dr}\{rf'(r)\}\cos\eta\right]$$

$$= \int ds \int ds' f(r)\cos\varepsilon, \ldots (12)$$

where the positive side of the surface is determined by the rule that a right-handed screw in that direction corresponds to the direction of the line integral round it.

We have proved that this result is the most general possible of its class.

Particular cases may be noted as follows:—

(i) Make the two circuits coincide.

(ii) Make the two open surfaces coincide, and we express the double surface integral by a double line integral round the contour. To avoid infinities, $f'(r)$ must not contain powers of r lower than the inverse first.

(iii) Make the surfaces plane, so that η is constant.

(iv) Make $$f'(r) = \frac{C}{r};$$

then

$$\iint dS \iint dS' \frac{\cos\theta\cos\theta'}{r^2} = -\tfrac{1}{2}\int ds \int ds' \log r \cos\varepsilon. \ldots (13)$$

The left-hand side is the expression for the illumination from S that is intercepted by S' when the brightness of S is unity; and it follows from elementary optical principles that this quantity must be expressible as a line integral round the contours of S and S'.

When S and S' coincide, we have

$$2\pi S = -\tfrac{1}{2}\int ds \int ds \log r \cos\varepsilon, \ldots\ldots\ldots\ldots (14)$$

true only when S is plane; for when S is not plane the real optical interpretation fails, the parts of the surface not being in full view of each other.

(v) Make $f(r) = \frac{C}{r}$, so that $f'(r) = -\frac{C}{r^2}$;

then

$$\iint dS \iint dS' \frac{\cos\eta - 3\cos\theta\cos\theta'}{r^3} = -\int ds \int ds' \frac{\cos\epsilon}{r}, \ldots (15)$$

which is Neumann's well-known expression for the mutual energy of two simple magnetic shells, or of two linear electric currents.

(vi) Make $$f'(r) = Cr;$$

then $$\int dS \int dS' \cos\eta = -\tfrac{1}{4}\int ds \int ds'\, r^2 \cos\epsilon, \ldots\ldots\ldots\ldots (16)$$

thus giving a double line integral form for $\int \Pi' dS$, where Π' denotes the area of the projection of S' on the tangent plane at dS. It was clear *à priori* that such a form must exist, for this integral depends only on S and the contours of S', while the other form $\int \Pi dS'$ shows that it depends only on the contour of S; thus the form of the function of r that multiplies $\cos\epsilon$ is all that remained *à priori* to be determined, and that might have been found from the simplest particular case.

When one surface S is plane, we have

$$S\Pi' = -\tfrac{1}{4}\int ds \int ds'\, r^2 \cos\epsilon, \ldots\ldots\ldots\ldots\ldots (17)$$

where Π' denotes the projection of S' on the plane of S.

Where S, S' coincide in one plane, we have

$$S^2 = -\tfrac{1}{4}\int ds \int ds\, r^2 \cos\epsilon. \ldots\ldots\ldots\ldots\ldots (18)$$

And comparing this with (iv) we deduce

$$(4\pi S)^2 = \{\int ds \int ds \log r \cos\epsilon\} = -4\pi^2 \int ds \int ds\, r^2 \cos\epsilon \ldots (19)$$

for any plane circuit; a striking result.

The theorems just given may be verified by direct integration when the surfaces are plane circles, [and (18) without much difficultly for the general surface]; by applying them to surfaces bounded by other curves, we obtain evaluations of a crop of definite integrals of somewhat unusual form.

III. If elements of three surfaces enter into a triple integral, the components of the elements of their three contours must enter, each linearly, into the corresponding line integral. The most general form of such line integral, independent of special coordinate systems, which gives a finite value when taken over complete circuits, is

$$\iiint \phi(r, r', r'') \begin{vmatrix} dx, & dy, & dz \\ dx', & dy', & dz' \\ dx'', & dy'', & dz'' \end{vmatrix},$$

where r, r', r'' are the mutual distances of the three elements of contour; and the determinant is equal to $3\Theta\, ds\, ds'\, ds''$, where

$$\Theta^2 = \sin\tfrac{1}{2}(a+b+c)\sin\tfrac{1}{2}(b+c-a)\sin\tfrac{1}{2}(c+a-b)\sin\tfrac{1}{2}(a+b-c),$$

a, b, c being the sides of the spherical triangle determined by the directions of ds, ds', ds''.

The integral may therefore by application of the method of II be expressed as a symmetrical triple surface integral; the general formulae are long, but the degenerate cases would probably be interesting.

Finally, there does not seem to be any reason why the considerations on which these theorems are founded should be restricted to the three dimensions x, y, z of ordinary space; but the more general results would probably be of only analytical interest.

ON THE ORDER OF PROOF OF THE PRINCIPAL EQUATIONS OF SPHERICAL TRIGONOMETRY.

By *M. Jenkins, M.A.*

THE principal formulas of spherical trigonometry are made to depend, wholly or partially, on three independent equations, which are of a less simple character than most of those which are derived from them; that is to say on

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

with the two other equations of the same form. Independent proofs are given of the more simple equations

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c};$$

but as these constitute only two independent equations, a third is needed for the complete investigation of the properties of a spherical triangle. I propose to take the equation

$$\frac{\sin(A+B)}{\sin C} = \frac{\cos a + \cos b}{1 + \cos c}$$

(which in the usual order would be obtained by the multiplication of the expressions of two of Gauss's theorems), give an independent proof of it, use the properties of colunar and of polar triangles to obtain the equations of similar form, and then show how these may be applied to prove other formulas.

As a lemma take the equation

$$\frac{\cos a + \cos b}{\cos m} = 2 \cos \tfrac{1}{2} c,$$

m being the median drawn from C to the mid-point of AB. Let ABC be a spherical triangle, D the mid-point of the arc AB, O the centre, $C\alpha$, $C\beta$, $C\delta$ the perpendiculars on OA, OB, OD respectively, $\delta\mu$, $\delta\nu$ perpendiculars on OA, OB; then because $C\alpha O$, $C\beta O$ and $C\delta O$ are right angles, the sphere on OC as diameter passes through α, β, δ. Therefore O, α, β, δ are on a circle; and because $\alpha O\delta = \beta O\delta$, $\delta\alpha = \delta\beta$, also $\delta\mu = \delta\nu$; hence $\mu\alpha = \nu\beta$; and μ, ν if distinct from α and β respectively (coinciding when $CA = CB$) must be on opposite sides of those points with regard to O, because the angles $O\alpha\delta$, $O\beta\delta$ are supplementary. Therefore

$$\frac{\cos a + \cos b}{\cos m} = \frac{O\alpha + O\beta}{O\delta} = \frac{2\, O\mu}{O\delta} = 2 \cos \tfrac{1}{2} c.$$

We may note that the same proof applies if D be not the mid-point of AB; except that, instead of $\delta\alpha$ being equal to $\delta\beta$, we have

$$\frac{\alpha\beta}{\sin c} = \frac{\alpha\delta}{\sin AD} = \frac{\beta\delta}{\sin BD},$$

also $$O\delta . \alpha\beta = O\beta . \alpha\delta + O\alpha . \beta\delta,$$

whence we have

$$\cos CD . \sin c = \cos a . \sin AD + \cos b . \sin BD.$$

Proceeding to the equation

$$\frac{\sin (A + B)}{\sin C} = \frac{\cos a + \cos b}{1 + \cos c},$$

let D be the mid-point of the arc AB, join CD; produce, making $DK = CD$, and join KA, KB so as to obtain the spherical rhomboid $ACBK$. In this, as in plane geometry, opposite sides and angles are equal, and alternate angles are equal; therefore the angle $CAK = A + B$. If $A + B < \pi$, CAK is a proper spherical triangle and denoting CD by m,

$$\frac{\sin CAK}{\sin ACK} = \frac{\sin 2m}{\sin AK} = \frac{\sin 2m}{\sin a},$$

$$\frac{\sin ACK}{\sin A} = \frac{\sin \frac{1}{2} c}{\sin m}, \text{ and } \frac{\sin A}{\sin C} = \frac{\sin a}{\sin c};$$

whence, by multiplication,

$$\frac{\sin(A+B)}{\sin C} = \frac{2\cos m}{2\cos\frac{1}{2}c} = \frac{2\cos m\cos\frac{1}{2}c}{2\cos^2\frac{1}{2}c}$$

$$= \frac{\cos a + \cos b}{1+\cos c}.$$

If $A+B>\pi$, CAK is not a proper spherical triangle; but since CK is also $>\pi$, if we cut off from it $CC'=\pi$, and treat the triangle KAC' as before, and note that $\sin(A+B)$ is of the same sign as $\sin CK$, we see that the previous equation still holds.

Next, by means of a colunar triangle A, a being unaltered, B changed into $\pi - B$, &c., we obtain

$$\frac{\sin(A-B)}{\sin C} = \frac{\cos b - \cos a}{1-\cos c};$$

from the polar triangle, changing $A = \pi - a$, &c., we have

$$\frac{\sin(a+b)}{\sin c} = \frac{\cos A + \cos B}{1-\cos C},$$

and from another colunar triangle,

$$\frac{\sin(a-b)}{\sin c} = \frac{\cos B - \cos A}{1+\cos C}, \text{ also } \frac{\sin A + \sin B}{\sin a + \sin b} = \frac{\sin C}{\sin c}.$$

Hence

$$\tan\tfrac{1}{2}(A+B) = \frac{\sin A+\sin B}{\cos A+\cos B} = \frac{\sin a+\sin b}{\sin c}\sin C.\frac{\sin c}{\sin(a+b)(1-\cos C)}$$

$$= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C,$$

$$\sin^2\tfrac{1}{2}(A+B) = \frac{(\cos B - \cos A)(\sin A + \sin B)}{2\sin(A-B)},$$

which, on substitution and reduction, gives

$$\frac{\cos^2\frac{1}{2}(a-b)}{\cos^2\frac{1}{2}c}\cos^2\tfrac{1}{2}C.$$

Similarly we may obtain the rest of Napier's analogies and Gauss's theorems, the sign in taking the square root being determined by the consideration that the greater side is opposite the greater angle in a colunar triangle as well as in the original triangle, that is $A+B-\pi$ of the same sign as $a+b-\pi$, as well as $A-B$ of the same sign as $a-b$.

CONTENTS.

The following papers have been received:

Major Allan Cunningham, "On the depression of differential equations."

Prof. W. Woolsey Johnson, "On the second solution of the differential equation of the hypergeometric series, and the series for K', E', &c., in Elliptic Functions."

Mr. F. Morley, "On plane cubics which inflect on crossing the asymptotes."

Articles for insertion will be received by the Editor, or by Messrs. W. Metcalfe and Son, Printing Office, Trinity Street, Cambridge

NOTICE.—A plate will be given whenever sufficient diagrams have been received.

No. CXCV.] NEW SERIES. [July, 1887.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XVII.—NO. 3.

MACMILLAN AND CO.
London and Cambridge.
1887.

W. METCALFE AND SON, PRINTERS, CAMBRIDGE.

Price—One Shilling.

To obtain $\cos C$,

$$\cos C = \frac{2\sin A\cos C}{2\sin A} = \frac{\sin B}{2\sin A}\,\frac{\sin(A+C)+\sin(A-C)}{\sin B}$$

$$= \frac{\sin b}{2\sin a}\left\{\frac{\cos a+\cos c}{1+\cos b} + \frac{\cos c-\cos a}{1-\cos b}\right\} = \frac{\cos c-\cos a\cos b}{\sin a\sin b}.$$

For right-angled triangles, if $C=\frac{1}{2}\pi$, $\sin A$ may be obtained direct from the sine-equation; $\cos c=\cos a\cos b$, by producing AC to A', so that $CA'=CA$; then CB being the median of the triangle ACA',

$$\cos c+\cos c = 2\cos b\cos a,$$

$$\frac{\cos A}{\sin B} = \cos a,$$

from the sine-equation by a modification of figure; or otherwise,

$$\frac{\cos A}{\sin B} = \frac{\sin(C+A)}{\sin B} = \frac{\cos c+\cos a}{1+\cos b} = \frac{\cos a\cos b+\cos a}{1+\cos b}$$

$$= \cos a,$$

$$\cos A = \frac{\sin(C+A)}{\sin C} = \frac{\sin B}{\sin C}\,\frac{\sin(C+A)}{\sin B}$$

$$= \frac{\sin b}{\sin c}\,\frac{\cos c+\cos a}{1+\cos b} = \frac{\sin b\cos a}{\sin c}$$

$$= \frac{\sin b}{\sin c}\,\frac{\cos c}{\cos b} = \frac{\tan b}{\tan c},$$

$$\tan A = \frac{\sin A}{\sin(C+A)} = \frac{\sin a}{\sin b}\,\frac{1+\cos b}{\cos c+\cos a} = \frac{\sin a}{\sin b\cos a} = \frac{\tan a}{\sin b},$$

$$\cot A\cot B = \frac{\sin(C+A)}{\sin B}\,\frac{\sin(C+B)}{\sin A} = \frac{\cos c+\cos a}{1+\cos b}\,\frac{\cos c+\cos b}{1+\cos a}$$

$$= \cos a\cos b = \cos c.$$

The equations for a quadrantal triangle may be found in a similar manner.

By eliminating $\cos c$ between

$$\frac{\sin(A+C)}{\sin A} = \frac{\sin b}{\sin a}\,\frac{\cos a+\cos c}{1+\cos b} \text{ and } \frac{\sin(A-C)}{\sin A} = \frac{\sin b}{\sin a}\,\frac{\cos c-\cos a}{1-\cos b}$$

we should obtain $\cos b\cos C = \cot a\sin b - \cot A\sin C$.

If the equation containing $\cos A+\cos B$ were deduced from that containing $\cos a+\cos b$ analytically, instead of by the use of the polar triangle, we should have to determine $\cos A$, $\cos B$ separately in terms of the sides. We may, however, prove the former equation independently thus:

In the figure to the lemma proved above the angular equation corresponding to the lemma is

$$\cos B \sin DCA - \cos A \sin BCD = \cos CDA \sin C,$$

the signs being most conveniently remembered by producing BA to C', and measuring the angles all one way, that is

$$\cos B \sin DCA + \cos CAC' \sin BCD = \cos CDA \sin BCA.$$

To prove this, draw the arc CN perpendicular to AB, CE perpendicular to ON; $E\alpha$, $E\beta$, $E\delta$ perpendicular to the planes OBC, OCA, OCD respectively, and EK perpendicular to OC. Then the five points E, δ, α, K, β are on a circle with EK for diameter, in a plane perpendicular to OK;

$$\begin{aligned} \alpha\beta : \beta\delta : \alpha\delta &= \sin \alpha E\beta : \sin \beta E\delta : \sin \alpha E\delta \\ &= \sin C \quad : \sin ACD : \sin BCD, \end{aligned}$$

also

$E\alpha = EC \cos CE\alpha = EC \cos B$; $E\beta = EC \cos CE\beta = EC \cos A$,
$E\delta = EC \cos CE\delta = EC \cos CDA$,

and $$E\alpha . \beta\delta = E\beta . \alpha\delta + E\delta . \alpha\beta,$$

whence $\cos B \sin DCA - \cos A \sin BCD = \cos CDA \sin C$.

Let ABC be a spherical triangle. In BC produced make $CE = CA$; join EA and draw the arc FCD bisecting the angle ECA, bisecting EA at right angles in F, and cutting AB produced in D. Then

$$\frac{\sin(a+b)}{\sin c} = \frac{\sin BE}{\sin BA} = \frac{\sin BAE}{\sin BEA} = \frac{\sin BAE}{\sin CAE} = \frac{\sin DAF}{\sin CAF};$$

but

$$\cos ADF \sin CAF - \cos AFC \sin CAD = \cos ACF \sin DAF,$$

and $$\cos AFC = 0,$$

therefore

$$\begin{aligned} \frac{\sin(a+b)}{\sin c} &= \frac{\cos ADF}{\cos ACF} = \frac{\cos ADF \sin ACB}{\cos ACF \sin ACB} \\ &= \frac{\cos B \sin ACD + \cos A \sin BCD}{\cos(\frac{1}{2}\pi - \frac{1}{2}C) \sin C} = \frac{(\cos B + \cos A)\cos\frac{1}{2}C}{\sin\frac{1}{2}C \sin C} \\ &= \frac{\cos B + \cos A}{1 - \cos C}. \end{aligned}$$

In a similar manner it could be proved that

$$\frac{\sin(a-b)}{\sin c} = \frac{\cos B - \cos A}{1 + \cos C}.$$

April, 11, 1887.

ON THE SECOND SOLUTION OF THE DIFFERENTIAL EQUATION OF THE HYPERGEOMETRIC SERIES, AND THE SERIES FOR K', E', &c., IN ELLIPTIC FUNCTIONS.

By Prof. *W. Woolsey Johnson.*

1. THE following solution of the case of failure of one of the ordinary solutions of a linear differential equation of the second order in series was suggested to me by Mr. Forsyth's solution of the corresponding case in Legendre's equation, *Messenger of Mathematics*, vol. XVI., p. 162. The form of solution is interesting as giving the series for the functions K', E' &c., in elliptic functions at once in the form given by Mr. Glaisher (*Camb. Phil. Proc.*, vol. v., p. 186), and accounting for the factors which he has called *the adjuncts*, occurring in the coefficients of the series (*Camb. Phil. Proc.*, vol. v., p. 240).

2. Denoting $x\dfrac{d}{dx}$ by ϑ, we suppose the differential equation to be

$$\phi(\vartheta)\,y - x^s\phi_1(\vartheta)\,y = 0 \;\ldots\ldots\ldots\ldots\ldots\; (1),$$

which is the most general form for which a relation exists between two consecutive coefficients of the series. The equation being of the second order we may assume

$$\phi\;(\vartheta) = (\vartheta - a)(\vartheta - b),$$

and

$$\phi_1(\vartheta) = p\,(\vartheta - c)(\vartheta - d),$$

where p is a positive or negative constant. We may also take the exponent s as unity; for putting $z = x^s$ and $z\dfrac{d}{dz} = \vartheta'$, we have

$$\vartheta' = x^s\frac{d}{sx^{s-1}dx} = \frac{1}{s}\vartheta,$$

and equation (1) becomes

$$\left(\vartheta' - \frac{a}{s}\right)\left(\vartheta' - \frac{b}{s}\right)y - pz\left(\vartheta' - \frac{c}{s}\right)\left(\vartheta' - \frac{d}{s}\right)y = 0,$$

which is of the form

$$(\vartheta - a)(\vartheta - b)\,y - px\,(\vartheta - c)(\vartheta - d)\,y = 0 \;\ldots\ldots\; (2).$$

We shall take equation (2) as the standard form, recollecting that when s is not unity the roots a, b and c, d are to be divided by s, and x^s substituted for x in the final result.

3. Putting in this equation

$$y = \Sigma_0^\infty A_r x^{m+r},$$

we have

$$\Sigma \{(m+r-a)(m+r-b) A_r x^{m+r} \\ - p(m+r-c)(m+r-d) A_r x^{m+r+1}\} = 0,$$

and since the coefficient of x^{m+r} must vanish,

$$(m+r-a)(m+r-b) A_r - p(m+r-c-1)(m+r-d-1) A_{r-1} = 0.$$

This gives the relation between consecutive coefficients,

$$A_r = p\frac{(m-c+r-1)(m-d+r-1)}{(m-a+r)(m-b+r)} A_{r-1};$$

and, when $r=0$,

$$(m-a)(m-b) A_0 = 0,$$

whence $m=a$ or $m=b$. Let $a \geqq b$, then taking $m=a$,

$$A_r = p\frac{(a-c+r-1)(a-d+r-1)}{r(a-b+r)} A_{r-1},$$

and the solution is

$$y = A_0 x^a \left[1 + \frac{(a-c)(a-d)}{1(a-b+1)} px \right. \\ \left. + \frac{(a-c)(a-c+1)(a-d)(a-d+1)}{1.2(a-b+1)(a-b+2)} (px)^2 + \ldots \right].$$

Again, interchanging a and b, the second solution is

$$y = B_0 x^b \left[1 + \frac{(b-c)(b-d)}{1(b-a+1)} px \right. \\ \left. + \frac{(b-c)(b-c+1)(b-d)(b-d+1)}{1.2(b-a+1)(b-a+2)} (px)^2 + \ldots \right].$$

To conform to the notation of the hypergeometric series let us put

$$\left.\begin{aligned} a-c &= \alpha \\ a-d &= \beta \\ a-b+1 &= \gamma \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

so that by the hypothesis made above $\gamma=1$; then denoting the complete integral by

$$y=A_0y_1+B_0y_2,$$

we have

$$y_1=x^a\left[1+\frac{\alpha.\beta}{1.\gamma}px+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}(px)^2+\dots\right]$$

$$=x^aF(\alpha,\beta,\gamma,px)\dots\dots\dots\dots\dots\dots\dots(4),$$

and

$$y_2=x^{a-\gamma+1}\left[1+\frac{(\alpha+1-\gamma)(\beta+1-\gamma)}{(2-\gamma).1}px\right.$$

$$\left.+\frac{(\alpha+1-\gamma)(\alpha+2-\gamma)(\beta+1-\gamma)(\beta+2-\gamma)}{(2-\gamma)(3-\gamma).1.2}(px)^2+\dots\right]\dots(5).$$

4. These series are identical when $\gamma=1$, and when $\gamma=q$, an integer greater than unity, the coefficient of $(px)^{q-1}$ in the second series is infinite; so that when γ is an integer these expressions fail to give two independent integrals.

To obtain a new integral in this case we put $\gamma=q-h$, and ultimately make $h=0$. The complete integral when $\gamma=q-h$ may be written

$$y=A_0y_1+B_0x^{a+1-\gamma}$$

$$\times\left[1+..+\frac{(\alpha+1-\gamma)\dots(\alpha+q-2-\gamma)(\beta+1-\gamma)\dots(\beta+q-2-\gamma)}{(2-\gamma)(3-\gamma)\dots(q-1-\gamma)(q-2)!}(px)^{q-2}\right]$$

$$+B_0\frac{(\alpha+1-\gamma)\dots(\alpha+q-1-\gamma)(\beta+1-\gamma)\dots(\beta+q-1-\gamma)}{(2-q+h)(3-q+h)\dots(-1+h)h(q-1)!}$$

$$\times p^{q-1}x^{a+h}\left[1+\frac{(\alpha+h)(\beta+h)}{(1+h)(\gamma+h)}px+\dots\right]\dots(6).$$

The first of the expressions multiplied by B_0 consists of the first $q-1$ terms of the general expression for y_2, equation (5), namely those which do not become infinite when $h=0$. Let these terms be denoted by T_{q-1}. If we denote the constant coefficient of the following expression by $\frac{B}{h}$, B will, when $h=0$, have the finite value

$$B=B_0\frac{(\alpha+1-q)\dots(\alpha-1)(\beta+1-q)\dots(\beta-1)}{(2-q)(3-q)\dots(-1)(q-1)!}p^{q-1}$$

$$=(-1)^qB_0\frac{(\alpha+1-q)\dots(\alpha-1)(\beta+1-q)\dots(\beta-1)}{(q-2)!\,(q-1)!}p^{q-1}\dots(7).$$

The factor $x^{a+h} = x^a x^h = x^a (1 + h \log x + \ldots)$, and finally the remaining factor in equation (6) is a function of h which we shall denote by $\psi(h)$, and which is such that by equation (4)

$$y_1 = x^a \psi(0).$$

The complete integral (6) may now be written

$$y = A_0 y_1 + B_0 T_{q-1} + \frac{B}{h}(1 + h \log x + \ldots) x^a [\psi(0) + h\psi'(0) + \ldots]$$

$$= Ay_1 + B_0 T_{q-1} + By_1 \log x + Bx^a \psi'(0) + \ \ldots\ldots\ldots\ldots\ldots \quad (8)$$

in which A is put for the constant $A_0 + \frac{B}{h}$, and the omitted terms are terms which vanish with h. It remains to express $\psi'(0)$ as a series in powers of x.

5. Let H_r denote the coefficient of $(px)^r$ in the series $\psi(h)$; that is, let

$$H_0 = 1, \text{ and } H_r = \frac{(\alpha+h)\ldots(\alpha+r-1+h)\,(\beta+h)\ldots(\beta+r-1+h)}{(1+h)\ldots(r+h)\,(\gamma+h)\ldots(\gamma+r-1+h)}.$$

Then $\psi(h) = \Sigma_0^\infty H_r (px)^r$, and $\psi'(h) = \Sigma_1^\infty \frac{dH_r}{dh}(px)^r$. Now

$$\frac{dH_r}{dh} = H_r \frac{d}{dh} \log H_r$$

$$= H_r \left[\frac{1}{\alpha+h} + \frac{1}{\alpha+1+h} + \ldots + \frac{1}{\alpha+r-1+h} + \frac{1}{\beta+h} + \ldots + \frac{1}{\beta+r-1+h}\right]$$

$$- H_r \left[\frac{1}{1+h} + \ldots + \frac{1}{r+h} + \frac{1}{\gamma+h} + \ldots + \frac{1}{\gamma+r-1+h}\right].$$

When $h = 0$, this becomes

$$H_r \Sigma_0^{r-1} \left[\frac{1}{\alpha+s} + \frac{1}{\beta+s} - \frac{1}{1+s} - \frac{1}{\gamma+s}\right]$$

in which H_r now denotes the coefficient of $(px)^r$ in the series $F(\alpha, \beta, \gamma, px)$, equation (4).

Thus

$$\psi'(0) = \Sigma_1^\infty \left\{H_r \Sigma_0^{r-1} \left[\frac{1}{\alpha+s} + \frac{1}{\beta+s} - \frac{1}{1+s} - \frac{1}{\gamma+s}\right](px)^r\right\};$$

and, writing the complete integral in equation (8) in the forms $y = Ay_1 + By_2$, we have for the two independent integrals when γ is an integral,

$$y_1 = x^a \left[1 + \frac{\alpha.\beta}{1.\gamma} px + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}(px)^2 + \ldots\right] \ldots (4),$$

and

$$y_2 = y_1 \log x$$
$$+(-1)^\gamma \frac{(\gamma-2)!\ (\gamma-1)!}{p^{\gamma-1}(\alpha+1-\gamma)\ldots(\alpha-1)(\beta+1-\gamma)\ldots(\beta-1)} T_{\gamma-1} + y' \ldots (9),$$

where

$$y' = x^a \left[\frac{\alpha.\beta}{1.\gamma}\left(\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{1} - \frac{1}{\gamma}\right) px + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}\right.$$
$$\left.\times \left(\frac{1}{\alpha} + \frac{1}{\alpha+1} + \frac{1}{\beta} + \frac{1}{\beta+1} - \frac{1}{1} - \frac{1}{2} - \frac{1}{\gamma} - \frac{1}{\gamma+1}\right)(px)^2 + \ldots\right]$$
$$\ldots\ldots\ldots (10),$$

and $T_{\gamma-1}$ denotes the sum of the first $\gamma - 1$ terms of y_2 in equation (5). When $\gamma = 1$ we have of course $T_0 = 0$, and when $\gamma = 2$, $T_1 = x^{a-\gamma+1} = x^{a-1}$.

Thus the second solution consists of three parts, the first of which is the product of the first solution by $\log x$, the second is a finite series beginning with $x^{a-\gamma+1}$ and ending with the power x^{a-1}, and the third is the secondary series y', which is the same as y_1 except that each coefficient is multiplied by a factor which we shall call its *adjunct*, consisting of the sum of the reciprocals of the factors in the numerator taken positively and of those in the denominator taken negatively. The first coefficient of y_1 which is 1 must be considered as having the adjunct zero.

6. The law of the adjuncts just stated is the same as that pointed out by Mr. Glaisher in the case of the series for K', E', &c., in the paper cited in § 1, although the notation differs from that of the hypergeometric series. The reason for this persistence of the law is readily explained as follows, and is illustrated in the examples given in the succeeding sections.

In the first place, if α and β are fractions having the same denominator m, say $\alpha = \frac{\alpha'}{m}$, $\beta = \frac{\beta'}{m}$, the coefficients in y_1,

$$\frac{\frac{\alpha'}{m}\cdot\frac{\beta'}{m}}{1.\gamma},\ \frac{\frac{\alpha'}{m}\left(\frac{\alpha'}{m}+1\right)\frac{\beta'}{m}\left(\frac{\beta'}{m}+1\right)}{1.2.\gamma(\gamma+1)}, \ldots$$

may be written in the form

$$\frac{\alpha'.\beta'}{m.m\gamma}, \quad \frac{\alpha'(\alpha'+m)\,\beta'(\beta'+m)}{m.2m.m\gamma(m\gamma+m)} \cdots.$$

If the adjuncts be now formed by the same law as before, each term of each will have one mth of its former value, and the result will be the value of $\frac{1}{m}y'$, so that the law holds good for the integral $\frac{1}{m}y_2$, when the coefficients are written in the new form.

Again, if we have occasion to introduce new factors into the numerators or denominators of the coefficients in y_1, so that, for instance, we write $\frac{a}{b}y_1$ for the first solution, then each adjunct if formed by the same law should contain the additional terms $\frac{1}{a}-\frac{1}{b}$. If then we take $\frac{a}{b}y_2+\left(\frac{1}{a}-\frac{1}{b}\right)y_1$ for the second integral, the law of the adjunct will still hold good.

7. We proceed to apply the formulae given above in the case of the differential equation satisfied by K and K', the independent variable being the modulus k. This equation is

$$k(1-k^2)\frac{d^2y}{dk^2}+(1-3k^2)\frac{dy}{dk}-ky=0.$$

Multiplying by k and putting $k\frac{d}{dk}=\vartheta$, whence

$$k^2\frac{d^2}{dk^2}=\vartheta(\vartheta-1),$$

this becomes

$$\vartheta^2y-k^2(\vartheta^2+2\vartheta+1)\,y=0;$$

and putting $x=k^2$ to reduce it to the standard form, equation (2), we have

$$\vartheta^2y-x(\vartheta+\tfrac{1}{2})^2y=0.$$

Here $p=1$, and

$$\begin{array}{lcl} a=0, & & a=0, \\ b=0, & \text{whence} & \alpha=\tfrac{1}{2}, \\ c=-\tfrac{1}{2}, & & \beta=\tfrac{1}{2}, \\ d=-\tfrac{1}{2}, & & \gamma=1. \end{array}$$

Equations (4) and (9) give then for our two integrals

$$y_1 = 1 + \frac{(\frac{1}{2})^2}{1^2} k^2 + \frac{(\frac{1}{2})^2 (\frac{3}{2})^2}{1^2.2^2} k^4 + \ldots,$$

$$y_2 = y_1 \log k^2 + \frac{(\frac{1}{2})^2}{1^2} (2 + 2 - 1 - 1) k^2$$

$$+ \frac{(\frac{1}{2})^2 (\frac{3}{2})^2}{1^2.2^2} (2 + 2 + \tfrac{2}{3} + \tfrac{2}{3} - 1 - 1 - \tfrac{1}{2} - \tfrac{1}{2}) k^4 + \ldots .$$

It is customary to write the first series, which is the value of $\dfrac{2K}{\pi}$, in the form

$$\frac{2K}{\pi} = y_1 = 1 + \frac{1^2}{2^2} k^2 + \frac{1^2.3^2}{2^2.4^2} k^4 + \ldots,$$

and accordingly we write the value of $\frac{1}{2}y_2$, which is

$$\tfrac{1}{2}y_2 = \tfrac{1}{2}y_1 \log k^2 + \frac{1^2}{2^2} (1 + 1 - \tfrac{1}{2} - \tfrac{1}{2}) k^2$$

$$+ \frac{1^2.3^2}{2^2.4^2} (1 + 1 + \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{4}) k^4 + \ldots,$$

in which the law of the adjuncts is followed. Now $K' = y_1 \log 4 - \frac{1}{2}y_2$ so that the law holds for the series for K' which Mr. Glaisher writes in the form

$$K' = \log \frac{4}{k} + \frac{1^2}{2^2} \left(\log \frac{4}{k} - \tfrac{2}{1} + \tfrac{2}{2}\right) k^2 + \frac{1^2.3^2}{2^2.4^2} \left(\log \frac{4}{k} - \tfrac{2}{1} + \tfrac{2}{2} - \tfrac{2}{3} + \tfrac{2}{4}\right) k^4 + \ldots$$

8. Consider next the equation satisfied by E and I', which is

$$k(1 - k^2) \frac{d^2y}{dk^2} + (1 - k^2) \frac{dy}{dk} + ky = 0,$$

or

$$\vartheta^2 y - k^2 (\vartheta^2 - 1) y = 0.$$

Putting $x = k^2$, this becomes

$$\vartheta^2 y - x (\vartheta^2 - \tfrac{1}{4}) y = 0,$$

where again $p = 1$, and

$$\begin{array}{lcl} a = 0, & & a = 0 \\ b = 0, & \text{whence} & \alpha = -\tfrac{1}{2}, \\ c = \tfrac{1}{2}, & & \beta = \tfrac{1}{2}, \\ d = -\tfrac{1}{2}, & & \gamma = 1. \end{array}$$

Thus $$y_1 = 1 + \frac{-\frac{1}{2}.\frac{1}{2}}{1.1} k^2 + \frac{-\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{3}{2}}{1.1.2.2} k^4 + \dots,$$

and

$$y_2 = y_1 \log k^2 + \frac{-\frac{1}{2}.\frac{1}{2}}{1.1}(-2+2-1-1)k^2$$
$$+ \frac{-\frac{1}{2}.\frac{1}{2}.\frac{1}{2}.\frac{3}{2}}{1.1.2.2}(-2+2+2+\tfrac{2}{3}-1-1-\tfrac{1}{2}-\tfrac{1}{2})k^4 + \dots .$$

But y_1 which is the value of $\frac{2E}{\pi}$ is usually written

$$\frac{2E}{\pi} = y_1 = 1 - \frac{1}{2^2}k^2 - \frac{1^2.3}{2^2.4^2}k^4 - \frac{1^2.3^2.5}{2^2.4^2.6^2}k^6 - \dots,$$

in which we have dropped from the numerators of the coefficients the factor 1 which corresponds to the first factor $(-\frac{1}{2})$ in the coefficients as first written. Accordingly if we write

$$\tfrac{1}{2}y_2 = \tfrac{1}{2}y_1 \log k^2 - \frac{1}{2^2}(-1+1-\tfrac{1}{2}-\tfrac{1}{2})k^2$$
$$- \frac{1^2.3}{2^2.4^2}(-1+1+1+\tfrac{1}{3}-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{4}-\tfrac{1}{4})k^4 - \dots,$$

the law of the adjunct is not followed because the term -1 in each adjunct is now superfluous; but adding y_1 we get rid of these terms and have the integral

$$y_1 + \tfrac{1}{2}y_2 = 1 + \tfrac{1}{2}y_1 \log k^2 - \frac{1}{2^2}(1-\tfrac{1}{2}-\tfrac{1}{2})k^2$$
$$- \frac{1^2.3}{2^2.4^2}(1+1+\tfrac{1}{3}-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{4}-\tfrac{1}{4})k^4 - \dots,$$

in which the law holds good. Now $-I'$ is found to be $y_1 \log 4 - (y_1 + \frac{1}{2}y_2)$, and thus satisfies the law of the adjuncts.

9. The equation satisfied by I and E' is, when $x = k'^2$,

$$(\vartheta^2 - \vartheta)y - x(\vartheta^2 - \tfrac{1}{4})y = 0,$$

in which

$$\begin{aligned} a &= 1, \\ b &= 0, \\ c &= \tfrac{1}{2}, \\ d &= -\tfrac{1}{2}, \end{aligned} \quad \text{whence} \quad \begin{aligned} a &= 1, \\ \alpha &= \tfrac{1}{2}, \\ \beta &= \tfrac{3}{2}, \\ \gamma &= 2. \end{aligned}$$

Hence

$$y_1 = k^2\left[1+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1.2}k^2+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.2.3}k^4+\dots\right],$$

and, from equation (9),

$$y_2 = y_1\log k^2+\frac{1}{-\frac{1}{2}\cdot\frac{1}{2}}+k^2\left[\frac{\frac{1}{2}\cdot\frac{3}{2}}{1.2}(2+\tfrac{2}{3}-1-\tfrac{1}{2})k^2+\dots\right].$$

Here, as before, $\frac{1}{2}y_2$ would follow the adjunct law when the coefficients are expressed as simple fractions; but the leading solution which is the value of $-\dfrac{2I}{\pi}$ is here $\frac{1}{2}y_1$, thus

$$-\frac{2I}{\pi}=\tfrac{1}{2}y_1=\frac{1}{2}k^2+\frac{1^2.3}{2^2.4}k^4+\frac{1^2.3^2.5}{2^2.4^2.6}k^6+\dots.$$

If now we write the value of $\frac{1}{4}y_2$ with corresponding coefficients, we have

$$\tfrac{1}{4}y_2=\tfrac{1}{4}y_1\log k^2-1+\frac{1^2.3}{2^2.4}(1+\tfrac{1}{3}-\tfrac{1}{2}-\tfrac{1}{4})k^4+\dots,$$

in which the law is not observed because, owing to the new factors, 1 in the numerator and 2 in the denominator, the additional terms $1-\frac{1}{2}$ are required in each adjunct. But adding one half of the value of $-\dfrac{2I}{\pi}$ to supply these terms, we have

$$\tfrac{1}{4}y_2+\tfrac{1}{4}y_1=\tfrac{1}{4}y_1\log k^2-1+\frac{1}{2}(1-\tfrac{1}{2})k^2$$
$$+\frac{1^2.3}{2^2.4}(1+1+\tfrac{1}{3}-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{4})k^4+\dots,$$

in which the law is followed. The value of E' is found to be $E'=\frac{1}{2}y_1\log 4-(\frac{1}{4}y_2+\frac{1}{4}y_1)$, and thus also satisfies the law of the adjuncts.

10. Of the seven examples given by Mr. Glaisher in the paper cited in § 1, we give but one more, and that on account of an apparent anomaly in one of the coefficients. The equation satisfied by $I+G$ and $E'+G'$ is, when $x=k^2$,

$$\vartheta(\vartheta-2)y-x(\vartheta-\tfrac{1}{2})^2y=0,$$

in which

$$\begin{array}{lcl} a=2, & & a=2, \\ b=0, & \text{whence} & \alpha=\tfrac{3}{2}, \\ c=\tfrac{1}{2}, & & \beta=\tfrac{3}{2}, \\ d=\tfrac{1}{2}, & & \gamma=3. \end{array}$$

Thus $$y_1=k^4\left[1+\frac{\frac{3}{2}\cdot\frac{3}{2}}{1.3}k^2+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3.4}k^4+\dots\right]$$

and $$y_2=y_1\log k^2-\frac{2}{-\frac{1}{2}\cdot\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}}\left[1+\frac{-\frac{1}{2}\left(-\frac{1}{2}\right)}{-1.1}k^2\right]$$

$$+k^4\left[\frac{\frac{3}{2}\cdot\frac{3}{2}}{1.3}\left(\tfrac{2}{3}+\tfrac{2}{3}-1-\tfrac{1}{3}\right)k^2+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3.4}\left(\tfrac{2}{3}+\tfrac{2}{5}+\tfrac{2}{3}+\tfrac{2}{5}-1-\tfrac{1}{2}-\tfrac{1}{3}-\tfrac{1}{4}\right)k^4+\dots\right],$$

the expression for $T_{\gamma-1}$ in equation (9) in this case containing two terms.

The value of $-\dfrac{2(I+G)}{\pi}$ is $\frac{1}{8}y_1$, thus

$$-\frac{2(I+G)}{\pi}=\tfrac{1}{8}y_1=\frac{1^2}{2.4}k^4+\frac{1^2.3^2}{2^2.4.6}k^6+\dots,$$

and if we write accordingly the value of $\frac{1}{16}y_2$ we have

$$\tfrac{1}{16}y_2=\tfrac{1}{16}y_1\log k^2-2\left(1-\tfrac{1}{4}k^2\right)+\frac{1^2.3^2}{2^2.4.6}\left(\tfrac{1}{3}+\tfrac{1}{3}-\tfrac{1}{2}-\tfrac{1}{6}\right)k^6+\dots$$

in which, in order to follow the law, the terms $1+1-\frac{1}{2}-\frac{1}{4}$ are needed in each adjunct. Adding therefore $\frac{5}{4}$ of the preceding series we have

$$\tfrac{1}{16}y_2+\tfrac{5}{32}y_1=\tfrac{1}{16}y_1\log k^2-2+\tfrac{1}{2}k^2$$
$$+\frac{1^2}{2.4}\left(1+1-\tfrac{1}{2}-\tfrac{1}{4}\right)k^4+\frac{1^2.3^2}{2^2.4.6}\left(1+1+\tfrac{1}{3}+\tfrac{1}{3}-\tfrac{1}{2}-\tfrac{1}{2}-\tfrac{1}{4}-\tfrac{1}{6}\right)k^6+\dots,$$

in which the law of the adjuncts holds for the coefficient of k^4 and higher powers, but not for the coefficient of k^2. But this term, as we have seen above, is part of the expression for T_2 and not of that for y'. The value of $E'+G'$ is

$$\tfrac{1}{8}y_1\log 4-\tfrac{1}{16}y_2-\tfrac{5}{32}y_1.$$

11. The solution given in equations (4) and (9) is readily applied also to cases in which the function $\phi_1(\vartheta)$ of equation (1) is of a degree lower than the second. Thus Bessel's equation is

$$x^2\frac{d^2y}{dy^2}+x\frac{dy}{dx}+(x^2-n^2)\,y=0,$$

or

$$(\vartheta^2-n^2)\,y+x^2y=0.$$

Putting z in place of x^2, this becomes (see § 2)

$$(4\vartheta^2-n^2)\,y+zy=0,$$

which may be written

$$\left(\vartheta^2-\frac{n^2}{4}\right)y+z\frac{1}{4cd}(\vartheta-c)(\vartheta-d)\,y=0,$$

where $c=\infty$ and $d=\infty$. Comparing with equation (2) we have

$$\begin{aligned} a&=\tfrac{1}{2}n, & \qquad\qquad & \alpha=\tfrac{1}{2}n,\\ b&=-\tfrac{1}{2}n, & & \alpha=-c=\infty,\\ c&=\infty, & \text{whence} \quad & \beta=-d=\infty,\\ d&=\infty, & & \gamma=n+1,\\ p&=-\frac{1}{4cd}, & & p=-\frac{1}{4\alpha\beta}. \end{aligned}$$

Hence, by equations (4) and (5), § 3,

$$y_1=z^{\frac{1}{2}n}\left[1-\frac{1}{1\,(n+1)}\,(\tfrac{1}{4}z)+\frac{1}{1.2\,(n+1)\,(n+2)}\,(\tfrac{1}{4}z)^2-\dots\right],$$

or

$$y_1=x^n\left[1-\frac{1}{1\,(n+1)}\,(\tfrac{1}{2}x)^2+\frac{1}{1.2\,(n+1)\,(n+2)}\,(\tfrac{1}{2}x)^4-\dots\right],$$

and

$$y_2=x^{-n}\left[1-\frac{1}{1\,(1-n)}\,(\tfrac{1}{2}x)^2+\frac{1}{1.2\,(1-n)\,(2-n)}\,(\tfrac{1}{2}x)^4-\dots\right].$$

When $n=0$ or a positive integer the second solution fails, and we have by equation (9), § 5,

$$y_2=y_1\log x^2-n!\,(n-1)!\,4^nT_n+y',$$

where T_n denotes the sum of the first n terms of the preceding series for y_2, and

$$y' = x^n \left[\frac{1}{1\,(n+1)} \left(1 + \frac{1}{n+1}\right) (\tfrac{1}{2}x)^2 \right.$$
$$\left. - \frac{1}{1.2(n+1)(n+2)} \left(1 + \tfrac{1}{2} + \frac{1}{n+1} + \frac{1}{n+2}\right) (\tfrac{1}{2}x)^4 + \ldots \right],$$

in which the terms in the adjuncts corresponding to the α- and β-factors vanish because these factors are infinite. This solution agrees with that of Hankel, referred to by Mr. Forsyth in the paper cited in § 1.

12. When each of the functions ϕ and ϕ_1 is of the second degree we can derive a series either in ascending or in descending powers of x. For example, Legendre's equation,

$$(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n+1)\,y = 0,$$

may be written

$$\vartheta(\vartheta - 1)\,y - x^2 (\vartheta - n)(\vartheta + n + 1)\,y = 0,$$

in which $p = 1$, and $s = 2$; thus

$$\begin{aligned} a &= \tfrac{1}{2}, & \qquad && a &= \tfrac{1}{2}, \\ b &= 0, & && \alpha &= \tfrac{1}{2}(1-n), \\ c &= \tfrac{1}{2}n, & \text{whence} && \beta &= \tfrac{1}{2}(2+n), \\ d &= -\tfrac{1}{2}(n+1), & && \gamma &= \tfrac{3}{2}, \end{aligned}$$

and by equations (4) and (5) the integrals are

$$y_1 = x \left[1 + \frac{(1-n)(2+n)}{2.3} x^2 + \frac{(1-n)(3-n)(2+n)(4+n)}{2.3.4.5} x^4 + \ldots \right],$$

and

$$y_2 = 1 + \frac{-n(1+n)}{1.2} x^2 + \frac{-n(2-n)(1+n)(3+n)}{1.2.3.4} x^4 + \ldots.$$

13. But if we write the equation in the form

$$(\vartheta - n)(\vartheta + n + 1)\,y - \frac{1}{x^2} \vartheta(\vartheta - 1)\,y = 0,$$

we may take $s = -2$, and therefore

$$\begin{array}{lcl} a = \frac{1}{2}(n+1), & & a = \frac{1}{2}(n+1), \\ b = -\frac{1}{2}n, & \text{whence} & \alpha = \frac{1}{2}(n+1), \\ c = 0, & & \beta = \frac{1}{2}(n+2), \\ d = -\frac{1}{2}, & & \gamma = \frac{1}{2}(2n+3), \end{array}$$

and the integrals are

$$y_1 = x^{-n-1}\left[1 + \frac{(n+1)(n+2)}{2(2n+3)}x^{-2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-4} + \dots\right],$$

and

$$y_2 = x^n\left[1 - \frac{n(n-1)}{2(2n-1)}x^{-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{-4} + \dots\right].$$

14. When $n = -\frac{1}{2}$, $\gamma = 1$, and the two series are identical; and when $2n$ is a positive odd integer γ is an integer greater than 1, so that the series y_2 contains infinite coefficients. In these cases, therefore, we have by equation (9)

$$y_2 = y_1 \log\frac{1}{x^2}$$

$$+(-1)^{n+\frac{1}{2}}\frac{(n-\frac{1}{2})!\ (n+\frac{1}{2})!}{-\frac{1}{2}n\frac{1}{2}(2-n)\dots\frac{1}{2}(n-1).\frac{1}{2}(1-n)\frac{1}{2}(3-n)\dots\frac{1}{2}n}T_{n+\frac{1}{2}} + y',$$

or, taking $\frac{1}{2}y_2$ as the integral, because, as explained in § 6, $\frac{1}{2}y'$ will follow the adjunct law when the coefficients are written as in the expression for y_1 above,

$$\tfrac{1}{2}y_2 = y_1 \log\frac{1}{x} - \frac{\{(2n-1)(2n-3)\dots 4.2\}^2(2n+1)}{\{n(n-2)(n-4)\dots(1-n)\}^2}T_{n+\frac{1}{2}} + \tfrac{1}{2}y',$$

where $T_{n+\frac{1}{2}}$ denotes the sum of the first $n+\frac{1}{2}$ terms of the preceding series for y_2, and

$$\tfrac{1}{2}y' = x^{-n-1}\left[\frac{(n+1)(n+2)}{2(2n+3)}\left(\frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{2} - \frac{1}{2n+3}\right)x^{-2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}(ad)\,x^{-4} + \dots\right],$$

where the symbol (ad) written after a coefficient denotes its adjunct. This solution is the same as that given by Mr. Forsyth in the paper before cited.

When $2n$ is a negative odd integer the solution may be found in like manner, or at once by putting $-n-1$ in place of n in the solution just found, the result being

$$y_1 = x^n \left[1 + \frac{-n(1-n)}{2(1-2n)} x^{-2} + \frac{-n(1-n)(2-n)(3-n)}{2.4(1-2n)(3-2n)} x^{-4} + \ldots\right],$$

and

$$\tfrac{1}{2} y_2 = y_1 \log \frac{1}{x} - \frac{\{(-2n-3)(-2n-5)\ldots 4.2\}^2 (-2n-1)}{\{(-n-1)(-n-3)\ldots(2+n)\}^2} T_{-n-\frac{1}{2}} + \tfrac{1}{2} y',$$

in which $T_{-n-\frac{1}{2}}$ denotes

$$x^{-n-1}\left[1 + \frac{(n+1)(n+2)}{2(2n+3)} x^{-2} + \ldots \right.$$
$$\left. + \frac{(n+1)(n+2)\ldots(-n-3)}{2.4\ldots(-2n-3)(2n+3)\ldots(-4)(-2)} x^{2n+3}\right],$$

and

$$\tfrac{1}{2} y' = x^n \left[\frac{-n(1-n)}{2(1-2n)}(ad)x^{-2} + \frac{-n(1-n)(2-n)(3-n)}{2.4(1-2n)(3-2n)}(ad)x^{-4} + \ldots\right].$$

15. The functions ϕ and ϕ_1 in equation (1) being supposed real, the case we are considering, namely that in which γ is an integer, cannot arise when a and b, the roots of ϕ, are imaginary; but c and d, the roots of ϕ_1, may be imaginary, and α and β will then take the forms

$$\alpha = \mu + i\nu, \qquad \beta = \mu - i\nu.$$

In this case equations (4) and (5) may be written in the forms

$$y_1 = x^a \left[1 + \frac{\mu^2 + \nu^2}{1.\gamma} px + \frac{(\mu^2+\nu^2)\{(\mu+1)^2+\nu^2\}}{1.2.\gamma(\gamma+1)} (px)^2 + \ldots\right] \quad \ldots\ldots(4'),$$

$$y_2 = x^{a+1-\gamma}\left[1 + \frac{(\mu+1-\gamma)^2+\nu^2}{(2-\gamma).1} px \right.$$
$$\left. + \frac{\{(\mu+1-\gamma)^2+\nu^2\}\{(\mu+2-\gamma)^2+\nu^2\}}{(2-\gamma)(3-\gamma).1.2}(px)^2 + \ldots\right] \ldots(5');$$

and for the case in which γ is an integer equation (9) becomes

$$y_2 = y_1 \log x$$
$$+ (-1)^\gamma \frac{(\gamma-1)!\,(\gamma-2)!}{p^{\gamma-1}\{(\mu+1-\gamma)^2+\nu^2\}\ldots\{(\mu-1)^2+\nu^2\}}\, T_{\gamma-1} + y' \ldots (9'),$$

where

$$y' = x^\epsilon \left[\frac{\mu^2+\nu^2}{1.\gamma}\left(\frac{2\mu}{\mu^2+\nu^2} - 1 - \frac{1}{\gamma}\right) px \right.$$
$$+ \frac{(\mu^2+\nu^2)\{(\mu+1)^2+\nu^2\}}{1.2.\gamma(\gamma+1)}\left(\frac{2\mu}{\mu^2+\nu^2} + \frac{2(\mu+1)}{(\mu+1)^2+\nu^2} - 1 - \frac{1}{2}\right.$$
$$\left.\left. - \frac{1}{\gamma} - \frac{1}{\gamma+1}\right)(px)^2 + \ldots\right] \ldots (10').$$

These forms, with a change of sign of ν^2, are also convenient when α and β are irrational real numbers.

16. The form of the solution when γ is an integer undergoes modification when either α or β is an integer less than γ. In the first place, suppose one of them to be a positive integer less than γ. We cannot now employ equation (9), because the coefficient of $T_{\gamma-1}$ is infinite; in fact, B in equation (7) is now zero, and the complete integral (8) reduces to

$$y = Ay_1 + B_0 T,$$

in which T is a finite series containing $\gamma - \alpha$ terms. It is to be noticed that, comparing this with the general integral,

$$y = A_0 y_1 + B_0 y_2,$$

T is not equivalent to y_2, but we have

$$y_2 = T + \frac{B}{B_0 h} y_1,$$

in which the coefficient of y_1 takes the indeterminate form, but has a determinate value when γ and α are functions of a single quantity and become integers simultaneously.

This case, in which the finite series is not the limiting value of the infinite series from which it is derived, occurs in the solution of Riccati's equation (see the Memoir by J. W. L. Glaisher, *Phil. Trans.*, 1881, p. 771).

17. In the next place, let α or β be zero or a negative integer, say $\alpha=-n$; then y_1 is a finite series of which the last term is

$$x^{a}\frac{-n(-n+1)\ldots(-1)\,\beta(\beta+1)\ldots(\beta+n-1)}{n!\,\gamma(\gamma+1)\ldots(\gamma+n-1)}(px)^n.$$

In this case, supposing γ an integer, the series y' in the second integral is not a finite series, although the series y_1 from which it is derived is finite. Denoting the term of y_1 just written by $N(px)^n$, the coefficient is

$$N=(-1)^n\frac{\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)},$$

or, when $n=0$, $N=1$.

The coefficient of the next term, that is to say of the first term in y_1 which vanishes, is

$$N\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)},$$

the corresponding adjunct in y' contains the term $\dfrac{1}{\alpha+n}$, which is infinite, the remaining terms being finite. Thus the entire coefficient of $x^a(px)^{n+1}$ in y' is

$$N\frac{\beta+n}{(n+1)(\gamma+n)}.$$

In like manner, every succeeding adjunct contains the same infinite term, and the entire coefficient in y' is the same that it would be in y_1 with the omission of the zero factor. Thus, in addition to the part of y' formed by means of adjuncts from the actual terms in y_1, we have, corresponding to the vanishing part of y_1, the infinite series

$$N\frac{(\beta+n)p^{n-1}}{(n+1)(\gamma+n)}x^{a+n+1}\left[1+\frac{1.(\beta+n+1)}{(n+2)(\gamma+n+1)}px\right.$$
$$\left.+\frac{1.2.(\beta+n+1)(\beta+n+2)}{(n+2)(n+3)(\gamma+n+1)(\gamma+n+2)}(px)^2+\ldots\right].$$

ON PLANE CUBICS WHICH INFLECT ON CROSSING THEIR ASYMPTOTES.

By *F. Morley*.

1. Let a nodal cubic inflect on crossing an asymptote.

Taking the node as origin, and the x-axis parallel to the asymptote, we have in the equation to the cubic

$$a = c = c_1 = c_2 = 0.$$

Hence for the Hessian (Salmon's *Curves*, § 218),

$$\begin{aligned}
A &= \;\; 2ma_2a_3 - b_1a_3^2,\\
B &= -bm^2 + 2mb_3b_1 - a_2b_3^2,\\
C &= \;\; C_1 = C_2 = 0,\\
3A_2 &= -ba_3^2 + m^2a_2 + 2a_2a_3b_3,\\
3A_3 &= \;\; m^2a_3 - b_3a_3^2,\\
3B_1 &= -2bma_3 + m^2b_1 + 2b_1b_3a_3,\\
3B_3 &= m^2b_3 - a_3b_3^2,\\
6M &= 2m^3 - 2mb_3a_3.
\end{aligned}$$

Hence $$(m^2 - a_3b_3)\,U - 3H = 0 \quad\text{..................(1)}$$

has only terms of the third degree, and therefore represents the 3 lines from the node to the inflexions.

Expressing that there is no term in y^3,

$$b\,(m^2 - a_3b_3) = 3\,(-bm^2 + 2mb_3b_1 - a_2b_3^2) \quad\text{...... (2).}$$

Also the asymptote is $3a_2y + 3a_3 = 0$, and the cubic meets the y-axis where $by + 3b_3 = 0$, and since these are to give the same value for y,

$$ba_3 = 3a_2b_3.$$

Putting $b = 1$, and substituting for a_3 in (2),

$$m^2 = -3m^2 + 6mb_1b_3,$$

therefore $$m = 0 \text{ or } 2m = 3b_1b_3.$$

With the second value U becomes

$$y^3 + 3a_2x^2y + 3b_1xy^2 + 9a_2b_3x^2 + 9b_1b_3xy + 3b_3y^2 = 0,$$

or $$(y + 3b_3)\,(y^2 + 3b_1xy + 3a_2x^2) = 0,$$

so that the cubic degenerates.

With the first value for m the cubic is

$$y\,(y^2+3b_1xy+3a_2x^2)+3b_3\,(y^2+3a_2x^2)=0 \;\ldots\; (3).$$

Equation (1) now becomes

$$-a_3b_3\,(y^3+3b_1xy^2+3a_2x^2y)-3\,(-b_1a_3^2x^3+(2a_2a_3b_3-a_3^2)\,x^2y$$
$$+\,2b_1b_3a_3xy^2-a_2b_3^2y^3)=0,$$

or $\quad b_3\,(b_1xy^2+a_2x^2y)-b_1a_3x^3+(2a_2b_3-a_3)\,x^2y+2b_1b_3xy^2=0,$

or $$3b_3xy^2=3a_2b_3x^3,$$

or $$y^2=a_2x^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (4).$$

Since for 3 real inflexions a_2 must be positive, write $a_2=a^2$. Then the lines to the other inflexions are $y=\pm ax$, so that they are harmonic to the axes.

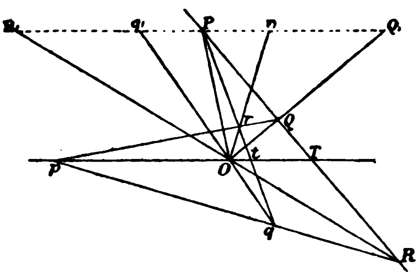
Substituting from (4) in (3),

$$y\,(y^2+3b_1xy+3y^2)+3b_3\,(y^2+3y^2)=0,$$

or $$4y+3b_1x+12b_3=0 \;\ldots\ldots\ldots\ldots\ldots\; (5),$$

which is the line of inflexions.

Let PQR be the inflexions, P being on the asymptote, and let the inflexional tangents form a triangle pqr. Let O be the node and let PQR and Pqr meet the x-axis in T and t.



It is calculated at once that the tangent at P is

$$3b_1x+y+3b_3=0 \;\ldots\ldots\ldots\ldots\ldots\; (6),$$

and at Q and R

$$3b_1x-8\,(y+3b_3)=\pm 9\,\frac{b_1}{a}\,y \ldots\ldots\ldots\ldots (7).$$

Hence it is manifest that p is on the x-axis; also putting $y=0$ in (5), (6), (7),

$$Op=2.OT=8.Ot.$$

Also from (6), (7), Or and Oq are

$$y = \pm 3ax \quad \text{.......................} \quad (8),$$

so that if OQ, OR, Oq, Or meet the asymptote in Q_1, R_1, q_1, r_1,

$$OQ_1 = OR_1 = 3\,Oq_1 = 3\,Or_1.$$

The equation to pQ_1 is

$$3b_1x - 8\,(y + 3b_2) = \frac{3b_1y}{a},$$

or

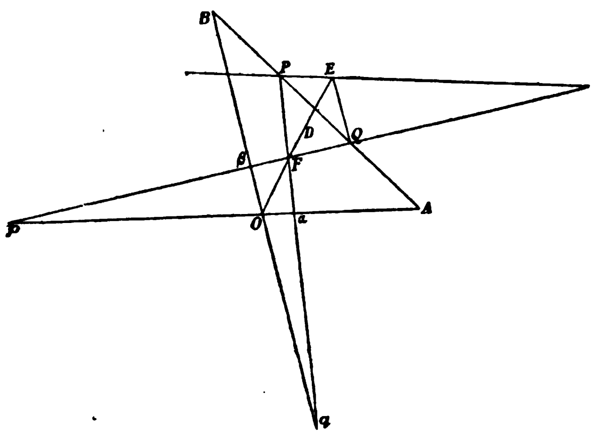
$$\frac{3b_1}{a}\,(3ax - y) = 2\,(3b_1x + 4y + 12b_2).$$

Hence pQ_1, Or and similarly pR_1, Oq meet on PQR. These are of course not all independent results.

2. If the cubic inflects on crossing a second asymptote, say at O, then Oq is parallel to the asymptote, so that $y + 3ax$ is a factor of $y^2 + 3b_1xy + 3a^2x^2$.

Hence the other factor is $y + ax$, and $3b_1 = 4a$. Hence $y + ax = 0$ meets the curve at infinity, or R is at infinity.

The metrical relations are now very simple.



Let OA, OB be parallel to the asymptotes at P, Q. Let $PFaq$, $QF\beta p$ be the inflexional tangents, then, since

$$Op = 2\,OA = 8\,Oa \quad \text{and} \quad Oq = 2\,OB = 8\,O\beta,$$

$a\beta$, pq, AB are parallel, and

$$pq = 2AB = 3AP = 3BQ,$$

therefore $$BP = PQ = QA;$$

hence pB, OQ are parallel, and also qA, OP.

Also if the asymptotes meet in E,

$$PD : DA = PE : OA = PQ : AB = 1 : 3,$$

therefore $$PD = \tfrac{1}{2}PQ = QD.$$

Also βQ is cut by OD in a ratio

$$= O\beta : QE = a\beta : PQ = \beta F : FQ,$$

therefore F lies on OE. And since

$ED : DO = EQ : BO = 1 : 3$ and $OF : FE = O\beta : QE = 3 : 4$

it follows that

$$DF : FO = OF : FE \text{ or } OF^2 = DF.EF.$$

Clearly all the lines in the figure, and all got by joining the points in the figure, are cut in very simple ratios.

Given O, P, Q, the construction for the asymptotes is : take on PQ points A, B such that $BP = PQ = QA$, then OA, OB are in the direction of the asymptotes. For the tangents : draw Bp parallel to OQ to meet AO. Then pQ is the tangent at Q.

The cubic cannot inflect on crossing all asymptotes unless each inflexion is at infinity, and the asymptotes are then the inflexional tangents.

3. A non-singular cubic has of course much more freedom. But if its inflects on crossing each asymptote the line of inflexions has an envelope which it is proposed to investigate.

Let $K \equiv x + y + z = 0$ be the line at infinity, $L \equiv ax + by + cz = 0$ be the line joining the points where the asymptotes meet the curve, and let $xyz = 0$ be the asymptotes.

Then the cubic is

$$LK^2 + 6mxyz = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9).$$

The first differential coefficients are

$$aK^2 + 2LK + 6myz, \ldots.$$

The second are, omitting a factor 2,

$$2aK + L, \ldots,$$

$$(b + c)K + L + 3mx, \ldots.$$

The Hessian formed from these last six works down to

$$\begin{aligned}&2K^2\{(a-b)(a-c)x+\ldots\}\\&+6mK\{(b+c)yz-ax^2+\ldots\}\\&+3mL(2yz-x^2+\ldots)\\&+18m^2xyz=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).\end{aligned}$$

Let $x=0$ and $L=0$,

therefore $2(y+z)^2\{(b-c)(b-a)y+(c-a)(c-b)z\}$

$$+6m(y+z)\{(b+c)yz-by^2-cz^2\}=0,$$

or writing, since $by+cz=0$, $y=-c$, $z=b$,

$$(b-c)^3\{-c(b-a)-b(c-a)\}+3m(b-c)\{-bc(b+c)-bc^2-cb^2\},$$

or $$(b-c)^2\{2bc-a(b+c)\}+6mbc(b+c)=0\ldots(11),$$

[$b-c=0$ would mean an inflexion at infinity].

Suppose two such conditions hold:

$$(a-c)^2\{2ca-b(c+a)\}+6mca(c+a)=0,$$

$$(a-b)^2\{2ba-c(b+a)\}+6mab(a+b)=0.$$

Subtracting and dividing by $c-b$,

$$bc(2a-b-c)+6ma(b+c)+6ma^2+3a^3-2a^2(2c+2b)$$
$$+2(b^2+bc+c^2)a-abc=0.$$

Let a, b, c be the roots of

$$x^3-px^2+qx-r=0 \quad\ldots\ldots\ldots\ldots\ldots (12).$$

Then since

$$3abc+6ma(a+b+c)+3a^3-4a^2(b+c)+2a(b^2+c^2)-bc(b+c)=0,$$

therefore $3r+6map+3a^3-4a^2(p-a)$

$$+2a(p^2-2q-a^2)-\frac{r}{a}(p-a)=0,$$

and if the 3 such conditions hold, a, b, c are given by

$$4rx+6mpx^2+5x^4-4px^3+2x^2(p^2-2q)-pr=0.$$

Dividing this by (12) the remainder is

$$3(p^2-3q+2mp)x^2+(qr-pq)x,$$

and therefore the only conditions which the constants must satisfy are

$$2m = \frac{3q - p^2}{p} = \frac{bc + ca + ab - a^2 - b^2 - c^2}{a + b + c},$$

and $$pq = 9r \text{ .... } (13),$$

or $$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 = 0.$$

Given the asymptotes, the envelope of the line of inflexions is the envelope of $ax + by + cz = 0$ subject to (13). It may of course be got by eliminating a, but the following symmetrical method gives the result in a better form. Equating differential coefficients with regard to a, b, c, and neglecting a constant,

$$x = q + (b + c)p - 9bc$$

$$= q + p^2 - pa - \frac{9r}{a},$$

or $$pa^2 - (p^2 + q - x)a + 9r = 0,$$

and $$a^3 - pa^2 + qa - r = 0.$$

From these we must eliminate a. Multiply the second by 9 and add:

$$9a^2 - 8pa - (p^2 - 8q - x) = 0,$$

therefore $$\{9(p^2 + q - x) - 8p^2\}\{(p^2 + q - x)(p^2 - 8q - x) + 72pr\} = \{p(p^2 - 8q - x) + 81r\}^2,$$

or $$-9x^3 + 18x^2(p^2 - 3q) - x(9p^4 + 18p^2q - 9.15q^2 + 8p^4q - 72q^3 = 0,$$

or $$x^3 - Px^2 + Qx - R = 0,$$

where $$P = 2(p^2 - 3q),$$

$$Q = p^4 + 2p^2q - 15q^2 = (p^2 - 3q)(p^2 + 5q),$$

$$R = \tfrac{8}{9}(p^4q - 9q^3),$$

therefore $$p^2 - 3q = \tfrac{1}{2}P,$$

$$p^2 + 5q = \frac{2Q}{P},$$

$$q(p^2 + 3q) = \tfrac{9}{8}R \cdot \frac{2}{P},$$

therefore $$8q = \frac{2Q}{P} - \tfrac{1}{2}P,$$

and $$4(p^2 + 3q) = \tfrac{1}{2}P + \frac{6Q}{P},$$

therefore
$$\left(\frac{2Q}{P}-\tfrac{1}{2}P\right)\left(\tfrac{1}{2}P+\frac{6Q}{P}\right)=72\,\frac{R}{P},$$

or
$$(4Q-P^2)(12Q+P^2)=288PR\,\ldots\ldots\ldots(14).$$

The envelope is therefore a quartic.

It meets the line at infinity, $P=0$, on $Q=0$, which is the minimum ellipse about the triangle.

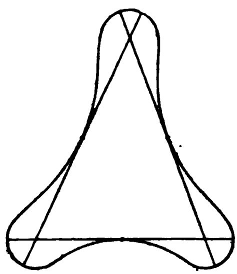
It meets the sides, $R=0$, on $4Q=P^2$, but this touches at the middle points, and also on $12Q+P^2=0$.

Hence it touches $x=0$ at the middle point and cuts its where

$$12yz+(y+z)^2=0,$$

or
$$y+7z=\pm 4\sqrt{3}z.$$

Hence the shape is as in the figure.



Bath College.

NOTE ON CERTAIN THEOREMS RELATING TO THE POLAR CIRCLE OF A TRIANGLE AND FEUERBACH'S THEOREM ON THE NINE-POINT CIRCLE.

By *Samuel Roberts.*

1. If with reference to a triangle the centres of the circumscribed and inscribed circles and the orthocentre are respectively O, I, P, and the radii of those circles and the polar circle are respectively R, r, ρ, then the following well-known relations hold:

$$OP^2=R^2+2\rho^2,\quad IP^2=2r^2+\rho^2.$$

The proof of the second equation has been found more difficult than that of the former,* while the comparison of the expressions indicates that they should both be deducible by methods closely analogous or differing correlatively.

In order to obtain such proofs I make use of two theorems which form part of the general theory of conics, but which, in relation to circles, strictly belong to the geometry of the front line and circle.

The theorems are:—

(I.) If a triangle be inscribed in one circle and self-conjugate with regard to another circle, any point on the first circle determines and is a vertex of a triangle inscribed in the first and self-conjugate with regard to the second circle.

(II.) If a triangle be circumscribed about one circle and self-conjugate with regard to another circle, any tangent to the first circle determines and is a side of a triangle circumscribed about the first and self-conjugate with regard to the second circle.

2. Consider two circles (C), (C'), whose centres are C, C' and whose radii are R, R'. Let the intersection of the circles be real.

Draw VW a common tangent touching (C) at V and (C') at W. Then, setting out with this tangent, V is its pole with respect to (C), and from V we draw VR, and again VW tangents to (C') (*i.e.*, VW counts as two coincident tangents). Then VW and VR constitute a limiting case of a triangle circumscribing the circle (C').

If this triangle be self-conjugate with respect to (C), VR must be perpendicular to CW.

Take this to be so, and let CW, $C'V$ intersect in S. Since $C'W$ is parallel to CV, we have

$$\angle SC'W = \angle C'SW = \angle CVS,$$

and S lies on the circle (C). Therefore $CW = R + R'$, and this gives

$$CC'^2 = (R+R')^2 + (R-R')^2 - R^2 = R^2 + 2R'^2.$$

Again, let L be an intersection of (C) and (C'), and draw LT touching (C') at L and meeting (C) again at T. Then

---

* See Mr. Leudesdorf's paper, "Proofs of Feuerbach's theorem," *Messenger of Mathematics*, vol. XIII., p. 116 *et seq.*

LT and the tangent to (C) at L constitute a limiting case of a triangle inscribed in (C).

If this triangle is self-conjugate with regard to (C'), the tangent to (C) at L is perpendicular to $C'T$. Let this be so; then, if $C'L$ meets (C) again in Q,

$$C'L = LQ;$$

and therefore

$$R^2 + CC'^2 = 2R^2 + 2R'^2, \text{ or } CC'^2 = R^2 + 2R'^2,$$

the same condition as before, showing, as the fact is, that if one triangle (and therefore an infinity of triangles) can be inscribed in one circle and self-conjugate with respect to a second circle, then one triangle (and therefore an infinity of triangles) can be circumscribed about the second circle and self-conjugate with regard to the first.

3. If we now return to the notation of (1), the foregoing results give at once

$$OP^2 = R^2 + 2\rho^2, \qquad IP^2 = \rho^2 + 2r^2.$$

And it has been often shown that these equations involve Feuerbach's Theorem.

In fact, writing K for the bisecting point of OP, we have

$$OI^2 + IP^2 = \tfrac{1}{2}(OP^2) + 2IK^2,$$

and also $$OI^2 = R^2 - 2Rr;$$

whence $$PK^2 = (\tfrac{1}{2}R - r)^2,$$

by substitution.

But K, $\frac{1}{2}R$ are respectively the centre and radius of the nine-point circle, which therefore touches the inscribed circle. A similar process applies to the escribed circles (W. F. Walker, *Quar. Journ. of Math.*, VIII., p. 47).

4. I will now justify, as shortly as I can, the assertion that theorems (I), (II) fall within the geometry of the point, line, and circle.

In Prof. Townsend's Treatise (vol. II. p. 168) and elsewhere we find:

THEOREM.—For every two self-reciprocal* triangles with respect to the same circle, every four of the six vertices connect and every four of the six sides intersect equianharmonically with the remaining two.

* Self-reciprocal = self-conjugate.

It is easy to establish also without direct reference to conics the following theorem:

"If A, B, C, D, E, F denote six points or lines, and that any four of the points connect equi-anharmonically with the remaining two, or any four of the lines intersect equi-anharmonically with the remaining two, and if any five of the points lie on, or any five of the lines touch a circle, the remaining point lies on, or the remaining line touches the same circle."

Now let ABC be a triangle self-conjugate with regard to a circle (S). Take any point D on the circle circumscribing the triangle ABC and draw the polar of D with respect to (S). Suppose that one of the intersections of this polar with the circle circumscribing the triangle is E. Then take F the pole of the line joining D, E with respect to (S). The triangle DEF will be self-conjugate with regard to (S), and by the above given theorems F lies on the circle through $ABCDE$, and must be the second intersection of the polar of D with that circle. This is theorem (I), and theorem (II) can be similarly proved.*

ON THE SYSTEM OF THREE CIRCLES WHICH CUT EACH OTHER AT GIVEN ANGLES AND HAVE THEIR CENTRES IN A LINE.

By Prof. *Cayley*.

In the system considered in the paper "System of equations for three given circles which cut each other at given angles," *Messenger*, t. XVII. pp. 18–21., we may consider the particular case where the centres of the circles are in a line. The condition in order that this may be so is obviously

$$\begin{vmatrix} \sin(A-\alpha)\cos F', & \sin(A-\alpha)\sin F', & \sin\alpha \\ \sin(B-\beta)\cos G', & \sin(B-\beta)\sin G', & \sin\beta \\ \sin(C-\gamma)\cos H', & \sin(C-\gamma)\sin H', & \sin\gamma \end{vmatrix} = 0,$$

that is,

$$\sin(B-\beta)\sin(C-\gamma)\sin\alpha\sin(G'-H')+\ldots=0;$$

---

* See Cremona's *Géométrie t. par Dewulf*, p. 224. The bare statement that if one triangle can be circumscribed about or inscribed in a circle and self-conjugate with regard to a second circle, an infinity of such triangles can be drawn, does not fully express the result.

or since $\sin(G'-H')$, $\sin(H'-F')$ $\sin(F'-G')$ are $=\sin A$, $\sin B$, $\sin C$ respectively, this is

$$\sin(B-\beta)\sin(C-\gamma)\sin A\sin\alpha+\ldots=0,$$

viz. this is

$$\frac{\sin A\sin\alpha}{\sin(A-\alpha)}+\frac{\sin B\sin\beta}{\sin(B-\beta)}+\frac{\sin C\sin\gamma}{\sin(C-\gamma)}=0,$$

or as this may also be be written

$$\frac{1}{\cot A-\cot\alpha}+\frac{1}{\cot B-\cot\beta}+\frac{1}{\cot C-\cot\gamma}=0.$$

But assuming this equation to be satisfied, it does not appear that there is any simple expression for the equation of the line through the three centres; nor would it be easy to transform the equations so as to have this line for one of the axes.

The case in question (which is a very important one from its connexion with Poincaré's theory of the Fuchsian functions) may be considered independently.

Taking the line of centres for the axis of x, and writing α, β, γ for the abscissæ of the centres, and P, Q, R for the radii, then the equations of the circles are

$$(x-\alpha)^2+y^2=P^2,$$
$$(x-\beta)^2+y^2=Q^2,$$
$$(x-\gamma)^2+y^2=R^2;$$

and then if the pairs of circles cut at the angles A, B, C respectively, we have

$$Q^2+2QR\cos A+R^2=(\beta-\gamma)^2,$$
$$R^2+2RP\cos B+P^2=(\gamma-\alpha)^2,$$
$$P^2+2PQ\cos C+Q^2=(\alpha-\beta)^2,$$

which are the equations connecting α, β, γ, P, Q, R. See the figure.

It is to be remarked in regard hereto that if A, B, C are used to denote the interior angles of the curvilinear triangle ABC, then the angles $\gamma A\beta$, $\alpha B\gamma$, $\beta C\alpha$ are $=\pi-A$, B, C respectively; whence if P, Q, R were used to denote the three radii taken positively, the first equation would be

$$Q^2+2QR\cos A+R^2=(\beta-\gamma)^2,$$

as above; but the other two equations would be

$$R^2 - 2RP\cos B + P^2 = (\gamma - \alpha)^2,$$

$$P^2 - 2PQ\cos C + Q^2 = (\alpha - \beta)^2;$$

hence, in order that the equations may be as above, it is necessary that P denote the radius of the circle, centre α, taken *negatively*; and it in fact appears that in a limiting case afterwards considered the value of P comes out negative. Similarly as regards the curvilinear triangle $AB'C'$; here A, $B(=B')$ and $C(=C')$ are the interior angles of the triangle; and the radius of the circle, centre α', must be regarded as negative.

Considering A, B, C as given, we have an equation between the radii P, Q, R. In fact this is at once obtained in the irrational form $\sqrt{(X)} + \sqrt{(Y)} + \sqrt{(Z)} = 0$, and proceeding to rationalise this, we obtain

$$-2\sqrt{(YZ)} = Y + Z - X,$$

that is $-\sqrt{\{(P^2 + 2PR\cos B + R^2)(P^2 + 2PQ\cos C + Q^2)\}}$

$$= P^2 + P(Q\cos C + R\cos B) - QR\cos A.$$

Hence, squaring and reducing, we find without difficulty

$$0 = Q^2R^2\sin^2 A + R^2P^2\sin^2 B + P^2Q^2\sin^2 C$$

$$+ 2P^2QR(\cos A + \cos B\cos C) + 2PQ^2R(\cos B + \cos C\cos A)$$

$$+ 2PQR^2(\cos C + \cos A\cos B,$$

or putting herein $P, Q, R = \dfrac{\sin A}{\xi}, \dfrac{\sin B}{\eta}, \dfrac{\sin C}{\zeta}$, this is

$$\left(1, 1, 1, \frac{\cos A + \cos B\cos C}{\sin B\sin C}, \frac{\cos B + \cos C\cos A}{\sin C\sin A},\right.$$

$$\left.\frac{\cos C + \cos A\cos B}{\sin A\sin B}\right)(\xi, \eta, \zeta)^2 = 0;$$

and it may be remarked that in this quadric form the three coefficients are each less than 1, or each greater than 1, according as $A + B + C > \pi$, or $A + B + C < \pi$.

Suppose 1°, $A + B + C > \pi$; the coefficients are here $= \cos\lambda$, $\cos\mu$, $\cos\nu$, the form is

$$(1, 1, 1, \cos\lambda, \cos\mu, \cos\nu \between \xi, \eta, \zeta)^2,$$

that is $(\xi + \eta\cos\nu + \zeta\cos\mu)^2 + (\eta^2\sin^2\nu + 2\eta\zeta\cos\lambda + \zeta^2\sin^2\mu)$,

namely this is

$$(\xi+\eta\cos\nu+\zeta\cos\mu)^2+\left\{\eta\sin\nu+\zeta\frac{\cos\lambda-\cos\mu\cos\nu}{\sin\nu}\right\}^2$$
$$+\zeta^2\left\{\sin^2\mu-\left(\frac{\cos\lambda-\cos\mu\cos\nu}{\sin\nu}\right)^2\right\};$$

where the last term is

$$=\frac{\zeta^2}{\sin^2\nu}\{\sin^2\mu\sin^2\nu-(\cos\lambda\cos\mu\cos\nu)^2\};$$

where the coefficient in $\{\quad\}$ is

$$=1-\cos^2\lambda-\cos^2\mu-\cos^2\nu+2\cos\lambda\cos\mu\cos\nu,$$

namely, substituting for $\cos\lambda$, $\cos\mu$, $\cos\nu$ their values, this is

$$=\frac{1}{\sin^2A\sin^2B\sin^2C}(1-\cos^2A-\cos^2B-\cos^2C-2\cos A\cos B\cos C)^2.$$

It thus appears that the form is the sum of three squares, and is thus constantly positive: it therefore only vanishes for imaginary values of the radii; or the case does not arise for any real figure.

Hence, 2°, if the figure be real, $A+B+C>\pi$, that is the sum of the angles of the curvilinear triangle is less than two right angles: the radii are connected as above by the equation

$$\left(1,\ 1,\ 1,\ \frac{\cos A+\cos B\cos C}{\sin B\sin C},\ \frac{\cos B+\cos C\cos A}{\sin C\sin A},\right.$$
$$\left.\frac{\cos C+\cos A\cos B}{\sin A\sin B}\right)\left(\frac{\sin A}{P},\ \frac{\sin B}{Q},\ \frac{\sin C}{R}\right)^2=0,$$

in which form the three coefficients are each greater than 1. Restoring therein ξ, η, ζ and regarding these as rectangular coordinates, the equation represents a real cone which might be constructed without difficulty, and then taking ξ, η, ζ as the coordinates of any particular point on the conical surface we have P, Q, $R=\dfrac{\sin A}{\xi}$, $\dfrac{\sin B}{\eta}$, $\dfrac{\sin C}{\zeta}$. Obviously, points on the same generating line of the cone give values of P, Q, R which differ in their absolute magnitudes only, their ratios being the same: the original equations in fact remain unaltered when P, Q, R, α, β, γ are each affected with any common factor.

Supposing P, Q, R taken so as to satisfy the equation in question, then taking the radicals

$$\sqrt{(Q^2+2QR\cos A+R^2)},\ \sqrt{(R^2+2RP\cos B+P^2)},$$
$$\sqrt{(P^2+2PQ\cos C+Q^2)},$$

with the proper signs we have a sum $=0$, and these give the values of $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$ respectively; and the construction of the figure would be thus completed.

I look at the question from a different point of view; taking Q, R, $\beta-\gamma$ such as to satisfy the first equation

$$Q^2+2QR\cos A+R^2=(\beta-\gamma)^2,$$

(that is starting from the two circles $(x-\beta)^2+y^2=Q^2$, $(x-\gamma)^2+y^2=R^2$ which cut each other at a given angle A), then the problem is to find a circle $(x-\alpha)^2+y^2=P^2$, cutting these at given angles C, B respectively; and to determine the coordinate of the centre α, and radius P, we have the remaining two equations

$$R^2+2RP\cos B+P^2=(\gamma-\alpha)^2,$$
$$P^2+2PQ\cos C+Q^2=(\alpha-\beta)^2.$$

namely, considering α, P as the coordinates of a point (in reference to the foregoing origin and axes), and for greater clearness writing $\alpha=\mathrm{x}$, $P=\mathrm{y}$ we have

$$\mathrm{y}^2+2\mathrm{y}R\cos B+R^2-(\mathrm{x}-\gamma)^2=0,$$
$$\mathrm{y}^2+2\mathrm{y}Q\cos C+Q^2-(\mathrm{x}-\beta)^2=0,$$

or as these may be written

$$(\mathrm{y}+R\cos B)^2-(\mathrm{x}-\gamma)^2=-R^2\sin^2 B,$$
$$(\mathrm{y}+Q\cos C)^2-(\mathrm{x}-B)^2=-Q^2\sin^2 C,$$

namely, the first of these equations denotes a rectangular hyperbola, coordinates of centre ($\mathrm{x}=\gamma$, $\mathrm{y}=-R\cos B$), transverse semi-axes $=R\sin B$; and the second of them a rectangular hyperbola coordinates of centre ($\mathrm{x}=\beta$, $\mathrm{y}=-Q\cos C$), transverse semi-axes $=Q\sin C$: as similar and similarly situate hyperbolas, these intersect in two points only; namely, the points are the intersections of either of them with the common chord

$$2\mathrm{y}\,(R\cos B-Q\cos C)+2\,(\gamma-\beta)\,\{\mathrm{x}-\tfrac{1}{2}(\gamma+\beta)\}+R^2-Q^2=0.$$

It is possible to construct a circle through the two points of intersection, and so to obtain these points as the intersections

of a line and circle; but the construction by the two rectangular hyperbolas is practically by no means an inconvenient one. I remark in passing that for a rectangular hyperbola the radius of curvature at the vertex is equal to the transverse semi-axis, and thus by drawing a small circular arc and by means of the asymptotes, we lay down a rectangular hyperbola graphically, without difficulty and with a fair amount of accuracy.

But the analytical solution may be carried somewhat further: we may without loss of generality write $\gamma=-\beta$, for this comes only to taking the origin midway between the centres of the circles β and γ: doing this and for greater simplicity writing also for the moment $Q\cos C=M$, $R\cos B=N$, the equations become

$$y^2+2yN+R^2-(x+\beta)^2=0,$$
$$y^2+2yM+Q^2-(x-\beta)^2=0,$$

where x is now the abscissa of the centre of the circle α (measured from the last-mentioned midway point) and y is the radius of this circle. We deduce

$$2(N-M)y+R^2-Q^2-4\beta x=0,$$

or say $$4\beta(x-\beta)=2(N-M)y+R^2-Q^2+4\beta^2,$$

and thence from the first equation multiplied by $16\beta^2$ we have

$$16\beta^2(y^2+2yN+R^2)-\{2(N-M)y+R^2-Q^2+4\beta^2\}^2=0,$$

that is

$$\begin{aligned}&4y^2\{4\beta^2-(N-M)^2\}\\&+4y\{4\beta^2(N+M)-(N-M)(R^2-Q^2)\}\\&+\quad\{8\beta^2(R^2+Q^2)-16\beta^4-(R^2-Q^2)^2\}=0,\end{aligned}$$

say this is $4y^2\mathfrak{A}+4y\mathfrak{B}+\mathfrak{C}=0$.

This gives

$$(2\mathfrak{A}y+\mathfrak{B})^2=\mathfrak{B}^2-\mathfrak{A}\mathfrak{C},$$

and we find without difficulty, restoring for M, N their values $Q\cos C$ and $R\cos B$,

$$\begin{aligned}\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}=4\beta^2\{&16\beta^4-8\beta^2(Q^2+R^2-2QR\cos B\cos C)\\&+(P^4-4P^3Q\cos B\cos C+2P^2Q^2(2\cos^2B+2\cos^2C-1)\\&\qquad-4PQ^3\cos B\cos C+Q^4)\},\end{aligned}$$

which is

$$=4\beta^2\{[4\beta^2-(Q^2-2QR\cos B\cos C+R^2)]^2-4Q^2R^2\sin^2B\sin^2C\}.$$

But we have

$$Q^2 + 2QR\cos A + R^2 = 4\beta^2,$$

and this equation thus becomes

$$\mathfrak{B}^2 - \mathfrak{A}\mathfrak{C} = 16\beta^2 Q^2 R^2 \{(\cos A + \cos B\cos C)^2 - \sin^2 B \sin^2 C\}$$

$$= 16\beta^2 Q^2 R^2 (-1 + \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C).$$

We have therefore

$$2\{4\beta^2 - (R\cos B - Q\cos C)^2\}\,\mathrm{y}$$
$$+ \{4\beta^2 (R\cos B + Q\cos C) - (R\cos B - Q\cos C)(R^2 - Q^2)\}$$
$$= \pm 4\beta QR\sqrt{\{-(1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C)\}},$$

$$4(R\cos B - Q\cos C)\,\mathrm{y} + R^2 - Q^2 = 4\beta\mathrm{x},$$

or completing the reduction by the substitution of the value of $4\beta^2$, this is

$$\mathrm{y}\{(Q^2\sin^2 C + R^2\sin^2 B) + 2QR(\cos A + \cos B\cos C)\}$$
$$+ QR\{Q(\cos B + \cos C\cos A) + R(\cos C + \cos A\cos B)\}$$
$$= \pm 4\beta QR\sqrt{\{-(1 - \cos^2 A - \cos^2 B - \cos^2 C - 2\cos A\cos B\cos C)\}},$$

viz. we have thus two values of the radius $\mathrm{y}\,(=P)$; and to each of these there corresponds a single value of the abscissa x, given by

$$4\beta\mathrm{x} = R^2 - Q^2 + 2(R\cos B - Q\cos C)\,\mathrm{y}.$$

The two values become equal if $A + B \pm C = \pi$; in this case the three circles meet in a pair of points $(\mathrm{x}_1, \mathrm{y}_1)$, $(\mathrm{x}_1, -\mathrm{y}_1)$. In fact, writing $A + B + C = \pi$, and thence

$$\cos A = -\cos(B + C), \ = -\cos B\cos C + \sin B\sin C, \ \&c.,$$

we find

$$\{Q^2\sin^2 C + 2QR(\cos A + \cos B\cos C) + R^2\sin^2 B\}\,\mathrm{y}$$
$$+ QR\{Q(\cos B + \cos C\cos A) + R(\cos C + \cos A\cos B)\} = 0,$$

that is

$$(Q\sin C + R\sin B)^2\,\mathrm{y} + QR(Q\sin C + R\sin B)\sin A = 0,$$

or throwing out the factor $(Q\sin C + R\sin B)$ this is

$$(Q\sin C + R\sin B)\,\mathrm{y} + QR\sin A = 0,$$

and we then have

$$4\beta x = R^2 - Q^2 - 2(R\cos B - Q\cos C)\frac{QR\sin A}{R\sin B + Q\sin C}$$

$$= \frac{1}{R\sin B + Q\sin B}\{(R\sin B + Q\sin C)(R^2 - Q^2)$$
$$-2(R\cos B - Q\cos C)\,QR\sin A\}.$$

The term in { } is here

$$\begin{aligned} & R^3(\sin B) \\ +\, & R^2Q(\ \ \sin C - 2\sin A\cos B) \\ +\, & RQ^2(-\sin B + 2\sin A\cos C) \\ +\, & \ \ Q^3(-\sin C), \end{aligned}$$

which is

$$\begin{aligned} = \quad & R^3(\ \ \sin B) \\ +\, & R^2Q(-\sin C + 2\sin B\cos A) \\ +\, & RQ^2(\ \ \sin B - 2\sin C\cos A) \\ +\, & RQ^3(-\sin C) \\ = \quad & (R^2 + Q^2 + 2RQ\cos A)(R\sin B - Q\sin C), \\ = \quad & 4\beta^2(R\sin B - Q\sin C), \end{aligned}$$

or finally

$$y = \frac{-QR\sin A}{R\sin B + Q\sin C},$$

$$x = \frac{\beta(R\sin B - Q\sin C)}{R\sin B + Q\sin C}.$$

In these equations y, x should be replaced by P, α respectively; and in obtaining them it was assumed that $\gamma = -\beta$; restoring the general values of β, γ the equations become

$$P = \frac{-QR\sin A}{R\sin B + Q\sin C},$$

$$\alpha - \tfrac{1}{2}(\beta + \gamma) = \frac{\tfrac{1}{2}(\beta - \gamma)(R\sin B - Q\sin C)}{R\sin B + Q\sin C},$$

viz. this last equation becomes

$$\alpha = \frac{\beta R \sin B + \gamma Q \sin C}{R \sin B + Q \sin C},$$

or say $\alpha (R \sin B + Q \sin C) - \beta R \sin B - \gamma Q \sin C = 0$,

which by means of the first equation becomes

$$\alpha \frac{QR}{P} \sin A + \beta R \sin B + \gamma Q \sin C = 0.$$

It thus appears that the two equations are

$$\frac{\sin A}{P} + \frac{\sin B}{Q} + \frac{\sin C}{R} = 0,$$

$$\frac{\alpha \sin A}{P} + \frac{\beta \sin B}{Q} + \frac{\gamma \sin C}{R} = 0,$$

viz. these equations, wherein $A + B + C = \pi$, belong to the case where the three circles intersect in the same pair of points; hence if the coordinates x, y refer to the points of intersection of the three circles, we have simultaneously the equations of the three circles, and the three equations which determine the angles at which they intersect, viz. we have the six equations

$$(x - \alpha)^2 + y^2 = P^2, \quad Q^2 + R^2 + 2QR \cos A = (\beta - \gamma)^2,$$

$$(x - \beta)^2 + y^2 = Q^2, \quad R^2 + P^2 + 2RP \cos B = (\gamma - \alpha)^2,$$

$$(x - \gamma)^2 + y^2 = R^2, \quad P^2 + Q^2 + 2PQ \cos C = (\alpha - \beta)^2,$$

viz. from these six equations, with the condition $A + B + C = \pi$, it must be possible to deduce the last-mentioned pair of equations.

In the general case, where $A + B + C < \pi$, and the three circles do not meet in a point, then taking the circles $(x - \beta)^2 + y^2 = Q^2$, $(x - \gamma)^2 + y^2 = R^2$ to be circles cutting each other at the angle A, or, what is the same thing, the values Q, R, β, γ to be such as to satisfy the relation

$$Q^2 + R^2 + 2QR \cos A = (\beta - \gamma)^2;$$

the two equations for the determination of the abscissa of the centre α, and the radius P of the remaining circle give, by what precedes

$$2\{(\beta-\gamma)^2-(R\cos B-Q\cos C)^2\}P$$
$$+\{(\beta-\gamma)^2(R\cos B+Q\cos C)-(R\cos B-Q\cos C)(R^2-Q^2)\}$$
$$=\pm 2(\beta-\gamma)QR\sqrt{\{-(1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C)\}},$$
$$4(R\cos B-Q\cos C)P+(R^2-Q^2)=(\beta-\gamma)(2\alpha-\beta-\gamma),$$

viz. we have thus the two circles $(x-\alpha)^2+y^2=P^2$, each of them cutting the circles $(x-\beta)^2+y^2=Q^2$, and $(x-\gamma)^2+y^2=R^2$ at the angles C, B respectively.

NOTE ON A THEOREM IN HIGHER ALGEBRA.

By *H. G. Dawson, B.A.*

THE theorem concerns the equivalence of the operators

$$p'\frac{d}{dp}+q'\frac{d}{dq}$$

and
$$a_0\frac{d}{da_1}+2a_1\frac{d}{da_2}+3a_2\frac{d}{da_3}+\&c.,$$

when applied to a function of the quantities $A_0, A_1, A_2 \ldots A_n$, where the transformation $x=pX+qY$, $y=p'X+q'Y$ transforms the binary form $(a_0a_1\ldots a_n)(x, y)^n$ into the form $(A_0, A_1, \ldots, A_n)(X, Y)^n$.

This equivalence, which is of importance in the theory of the covariants of the form, is proved somewhat tediously in Faa-de-Bruno's *Theorie des formes binaires*, p. 189, the equivalence having been previously stated in Section 77.

The proof now given seems interesting, not only as being expeditious, but also as pointing out very precisely the *raison d'être* of the equivalence in question, namely, that it is a necessary consequence of the linear character of the transformation.

The proof in question is as follows:

Let us suppose that the transformation

$$x=(p+\lambda p')X+(q+\lambda q')Y, \quad y=p'X+q'Y$$

is made, then we get a new form whose coefficients are

$$A_r+\lambda\left(p'\frac{dA_r}{dp}+q'\frac{dA_r}{dq}\right),$$

λ being supposed to be indefinitely small.

Now, owing to the linear character of the transformation, we might have proceeded in a different manner, thus, we might have put

$$x = u + \lambda v, \quad y = v,$$

and afterwards transform by the substitution

$$u = pX + qY, \quad v = p'X + q'Y.$$

The first of these two transformations gives us a form whose coefficients are a_0, $a_1 + 2\lambda a_2$, $a_2 + 3\lambda a_3$, &c., supposing λ^2 to be inappreciable, whilst the second gives $(A_0', A_1', \ldots, A_n')(X, Y)^n$, where A_0', A_1', ..., A_n' are just the same functions of a_0, $a_1 + 2\lambda a_2$, $a_2 + 3\lambda a_3$, &c., that A_0, $A_1 \ldots A_n$ are of $a_0, a_1 \ldots a_n$.

$$\text{Hence } A_r' = A_r + \lambda \left\{ a_1 \frac{dA_r}{da_0} + 2a_2 \frac{dA_r}{da_1} + 3a_3 \frac{dA_r}{da_2} + \ldots \right\}.$$

But, remembering our former view of the transformation,

$$A_r' = A_r + \lambda \left\{ p' \frac{dA_r}{dp} + q' \frac{dA_r}{dq} \right\},$$

we have the theorem in question, for

$$\left(p' \frac{d}{dp} + q' \frac{d}{dq}\right) F(A_0, A_1, \ldots A_n) = \frac{\partial F}{\partial A_0} \delta_1 A_0 + \ldots \frac{\partial F}{\partial A_r} \delta_1 A_r + \ldots,$$

where $$p' \frac{\partial}{\partial p} + q' \frac{\partial}{\partial q} = \delta_1.$$

The proof is easily applied to other forms; take for an example the ternary n^{ic} and write it as follows:

$$a_{0,0} x^n + n x^{n-1} (a_{1,0} y + a_{1,1} z) + \frac{n(n-1)}{1.2} (a_{2,0} y^2 + 2a_{2,1} yz + a_{2,2} z^2) x^{n-2}$$

$$+ \&\text{c.} + \frac{n!}{\rho!(n-\rho)!} \{a_{\rho,0} y^\rho + \rho a_{\rho,1} y^{\rho-1} z + \ldots\} x^{n-\rho} + \&\text{c.} \equiv (a)(xyz)^n.$$

Suppose that when we make the transformation

$$x = \lambda_1 X + \mu_1 Y + \nu_1 Z,$$
$$y = \lambda_2 X + \mu_2 Y + \nu_2 Z,$$
$$z = \lambda_3 X + \mu_3 Y + \nu_3 Z,$$

we obtain $$(A)(X, Y, Z)^n.$$

Let us now make the tranformation

$$x=(\lambda_1+\theta\lambda_2+\phi\lambda_3)X+(\mu_1+\theta\mu_2+\phi\mu_3)Y+(\nu_1+\theta\nu_2+\phi\nu_3)Z,$$
$$y=\lambda_2X+\mu_2Y+\nu_2Z,$$
$$z=\lambda_3X+\mu_3Y+\nu_3Z,$$

we thus obtain a form whose coefficients are

$$A_{\rho,\sigma}+(\theta\lambda_2+\phi\lambda_3)\frac{dA_{\rho,\sigma}}{d\lambda_1}+(\theta\mu_2+\phi\mu_3)\frac{dA_{\rho,\sigma}}{d\mu_1}+(\theta\nu_2+\phi\nu_3)\frac{dA_{\rho,\sigma}}{d\nu_1};$$

say $$A_{\rho\,\sigma}+\theta\bar{\delta}_2A_{\rho,\sigma}+\phi\bar{\delta}_3A_{\rho,\sigma}$$

where $$\bar{\delta}_2=\lambda_2\frac{d}{d\lambda_1}+\mu_2\frac{d}{d\mu_1}+\nu_2\frac{d}{d\nu_1},\ \bar{\delta}_3=\&c.;$$

again, the transformation could have been performed in another order, namely, by first using the substitution

$$x=U+\theta V+\phi W,$$
$$y=V,$$
$$z=W,$$

and afterwards putting

$$U=\lambda_1X+\mu_1Y+\nu_1Z,$$
$$V=\lambda_2X+\mu_2Y+\nu_2Z,$$
$$W=\lambda_3X+\mu_3Y+\nu_3Z.$$

Now if we make the first transformation, we easily see that the coefficient $a_{\rho,\sigma}$ changes to

$$a_{\rho,\sigma}+(\rho-\sigma)\theta a_{\rho-1,\sigma}+\sigma\phi a_{\rho-1,\sigma-1}.$$

[This is easily obtained, for θ and ϕ are of course course supposed to be very small though perfectly arbitrary; we are therefore at liberty to neglect all terms higher than the first (in θ and ϕ) of such a term as

$$(U+\theta V+\phi W)^{\kappa}].$$

Resuming, if we perform the second transformation we obtain a form $(A')(XYZ)^n$, whose coefficients are the same functions of the quantities

$$a_{\rho,\sigma}+(\rho-\sigma)\theta a_{\rho-1,\sigma}+\sigma\phi a_{\rho-1,\sigma-1},$$

that A_{00}, A_{10}, ... &c. are of $a_{\rho,\sigma}$, &c., and therefore

$$A'_{\rho,\sigma}=A_{\rho,\sigma}+\Sigma\Sigma\left\{\theta(p-q)a_{p-1,q}+\phi q a_{p-1,q-1}\right\}\frac{dA_{\rho,\sigma}}{da_{p,q}},$$

where the Σ applies to all such values of $a_{p,q}$ as can enter into $A_{\rho,\sigma}$.

But we obtained before the equation

$$A'_{\rho,\sigma} = A_{\rho,\sigma} + \theta\bar{\delta}_2 A_{\rho,\sigma} + \phi\bar{\delta}_3 A_{\rho,\sigma}.$$

Hence we have an equivalence betwen the operators

$$\lambda_2 \frac{d}{d\lambda_1} + \mu_2 \frac{d}{d\mu_1} + \nu_2 \frac{d}{d\nu_1} \text{ and } \Sigma\Sigma\,(p-q)\,a_{p-1,q}\frac{d}{da_{p,q}},$$

and between

$$\lambda_3 \frac{d}{d\lambda_1} + \mu_3 \frac{d}{d\mu_1} + \nu_3 \frac{d}{d\nu_1} \text{ and } \Sigma\Sigma q a_{p-1,q-1}\frac{d}{da_{p,q}},$$

for this reason, that θ and ϕ are perfectly arbitrary.

By writing the form in powers of y, we can obtain two operators which will be equivalent to

$$\lambda_1 \frac{d}{d\lambda_2} + \mu_1 \frac{d}{d\mu_2} + \nu_1 \frac{d}{d\nu_2} \text{ and } \lambda_3 \frac{d}{d\lambda_2} + \mu_3 \frac{d}{d\mu_2} + \nu_3 \frac{d}{d\nu_2};$$

similarly we can express the operators

$$\lambda_1 \frac{d}{d\lambda_3} + \mu_1 \frac{d}{d\mu_3} + \nu_1 \frac{d}{d\nu_3} \text{ and } \lambda_2 \frac{d}{d\lambda_3} + \mu_2 \frac{d}{d\mu_3} + \nu_2 \frac{d}{d\nu_3}$$

in terms of the coefficients.

The invariants of the form satisfying the six equations $\delta_2' = 0$, $\delta_3' = 0$, $\delta_1'' = 0$, $\delta_3'' = 0$, $\delta_1''' = 0$, $\delta_2''' = 0$ are the common solutions of the equations thus found.

If we take the case of the quadric form $(a, b, c, f, g, h)(xyz)^2$ our equations are

$$a\frac{d}{dg} + h\frac{d}{df} + 2g\frac{d}{dc},\; g\frac{d}{df} + 2h\frac{d}{db} + a\frac{d}{dh}.$$

Interchange x, y and we get

$$b\frac{d}{df} + h\frac{d}{dg} + 2f\frac{d}{dc},\; f\frac{d}{dg} + 2h\frac{d}{da} + b\frac{d}{dh}.$$

Interchange y, z,

$$c\frac{d}{df} + g\frac{d}{dh} + 2f\frac{d}{db},\; f\frac{d}{dh} + 2g\frac{d}{da} + c\frac{d}{dg},$$

and so on for other forms.

Christ's College, Cambridge,
July 30, 1887.

ON SYSTEMS OF RAYS.

By Prof. *Cayley*.

SIR W. R. HAMILTON'S Memoir, "Theory of Systems of Rays" (I do not at present consider the three Supplements), dated June, 1827, is printed *Trans. R. I. Acad.*, vol. XV. (1828), pp. 69–174. There is one page of Introduction, and pp. 70 to 80, an elaborate Contents. Part First: On ordinary systems of Reflected Rays. Part Second: On ordinary systems of Refracted Rays. Part Third: On extraordinary systems and systems of Rays in general. But only the First Part was published. This is considered under the headings: (I) Analytic Expressions of the Law of Ordinary Reflexion. (II) Theory of Focal Mirrors. (III) Surfaces of Constant Action. (IV) Classification of Systems of Rays. (V) On the Pencils of a Reflected System. (VI) On the Developable Pencils, the Two Foci of a Ray and the Caustic Curves and Surfaces. (VII) Lines of Reflection on a Mirror. (VIII) On Osculating Focal Mirrors. (IX) On Thin and Undevelopable Pencils. (X) On the Axes of a Reflected System. (XI) Images Formed by Mirrors. (XII) Aberrations. (XIII) Density. And we have, p, 174, a "Conclusion to the First Part," wherein this first part is described as "an attempt to establish general principles respecting the system of rays produced by the ordinary reflexion of light at any mirror or combination of mirrors shaped and placed in any manner whatever; and to show that the mathematical properties of such a system may all be deduced by analytic methods from the form of ONE CHARACTERISTIC FUNCTION: as in the application of Analysis to Geometry, the properties of a plane curve or of a curve surface may all be deduced by uniform methods from the form of the function which characterises its equation."

The foregoing headings (I) to (V) may be regarded as containing the general theory, and the remaining ones (VI) to (XIII) as containing applications and developments.

I remark on the theory as follows:

Considering a congruence or doubly infinite system of lines (or say of rays), suppose that for any particular ray the cosine-inclinations are α, β, γ ($\alpha^2+\beta^2+\gamma^2=1$), and that the coordinates of a point on the ray are (x, y, z). We may look at the system in two ways:

1°. The rays are considered as emanating from the points of a surface: here, if (x, y, z) are considered as belonging to

a point on the surface, then z is a given function of (x, y) (or more generally x, y, z are given functions of two arbitrary parameters u, v); and to determine the congruence we must have α, β, γ each of them a given function of (x, y) or of (u, v), such that identically $\alpha^2+\beta^2+\gamma^2=1$, but with no other condition as to the form of the functions.

2°. The rays may be considered irrespectively of any surface: here (α, β, γ) are each of them a given function of (x, y, z), such that always $\alpha^2+\beta^2+\gamma^2=1$; but there are further conditions, viz. it is assumed that we have one and the same ray, whatever is the point (x, y, z) on the ray; or what is the same thing (using ρ to denote an arbitrary distance), that α, β, γ regarded as functions of x, y, z remain unaltered by the change of x, y, z into $x+\rho\alpha$, $y+\rho\beta$, $z+\rho\gamma$ respectively; this implies that we have

$$(\alpha\delta_x+\beta\delta_y+\gamma\delta_z)\,\alpha,\ \ (\alpha\delta_x+\beta\delta_y+\gamma\delta_z)\,\beta,\ \ (\alpha\delta_x+\beta\delta_y+\gamma\delta)\,\gamma$$

each $=0$; and, conversely, it may be shown that when these relations are satisfied then α, β, γ remain unaltered by the change in question.

The equation $\alpha^2+\beta^2+\gamma^2=1$, gives

$$\alpha\delta_x\alpha+\beta\delta_x\beta+\gamma\delta_x\gamma,\quad \alpha\delta_y\alpha+\beta\delta_y\beta+\gamma\delta_y\gamma,\quad \alpha\delta_z\alpha+\beta\delta\,\beta+\gamma\delta\,\gamma$$

each $=0$; and combining with the last-mentioned system of equations, it follows that

$$\delta_z\beta-\delta_y\gamma,\quad \delta_x\gamma-\delta_z\alpha,\quad \delta_y\alpha-\delta_x\beta$$

are proportional to α, β, γ; or say $=k\alpha, k\beta, k\gamma$ respectively.

Hamilton considers whether the rays are such that they are cut at right angles by a surface; supposing this is so (say in this case the rays are a rectangular, or orthotomic system, or that they are the normals of a surface), then if x, y, z refer to the point on the surface, we must have

$$\alpha\,dx+\beta\,dy+\gamma\,dz=0;$$

this implies

$$\alpha\,(\delta_z\beta-\delta_y\gamma)+\beta\,(\delta_x\gamma-\delta_z\alpha)+\gamma\,(\delta_y\alpha-\delta_x\beta)=0,$$

a condition which, by what precedes, is $k\,(\alpha^2+\beta^2+\gamma^2)=0$; viz. we must have $k=0$, and therefore

$$\delta_z\beta-\delta_y\gamma,\quad \delta_x\gamma-\delta_z\alpha,\quad \delta_y\alpha-\delta_x\beta$$

each $=0$; that is, $\alpha\,dx+\beta\,dy+\gamma\,dz$ must be a complete differential, say it is $=dV$. And we then have $V=c$, the equation of a system of parallel surfaces each of them cutting

the rays at right angles. Evidently $\alpha, \beta, \gamma = \delta_x V, \delta_y V, \delta_z V$ respectively, and the function V satisfies the partial differential equation

$$(\delta_x V)^2 + (\delta_y V)^2 + (\delta_z V)^2 = 1.$$

Hamilton in effect considers only systems of rays of the form in question, viz. those which are the normals of a surface (or, what is the same thing, the normals of a system of parallel surfaces), and it is such a system which is said to have the characteristic function V. It is shown that a system of rays originally of this kind remains a system of this kind after any number of reflexions (or ordinary refractions); in particular if the rays originally emanate from a point, then, after any number of reflexions at mirrors of any form whatever, they are a system of rays cut at right angles by a surface. And moreover, there is given for the surface a simple construction, viz. starting from any surface which cuts the rays at right angles, and measuring off on the path of each ray (as reflected at the mirror or succession of mirrors) one and the same arbitrary distance, we have a set of points forming a surface which cuts at right angles the system of rays as reflected at the mirror or last of the mirrors.

The ray-systems considered by Hamilton are thus the normals of a surface $V - c = 0$, and a large part of the properties of the system are thus included under the known theory of the normals of a surface; it may be remarked that the analytical formulæ are somewhat simplified by the circumstance that V instead of being (as usual) an arbitrary function of (x, y, z) is a function satisfying the partial differential equation $(\delta_x V)^2 + (\delta_y V)^2 + (\delta V)^2 = 1$. In particular we have the theorem that each ray is intersected by two consecutive rays in foci which are the centres of curvature of the normal surface; the intersecting rays are rays proceeding from the curves of curvature of the normal surface, &c. There are other properties easily deducible from, but not actually included in, the theory of the normals; for instance, the intersecting rays aforesaid are rays proceeding from certain curves on the mirror, analogous to, but which obviously are not, the curves of curvature of the mirror. The natural mode of treatment of this part of the theory is to regard the rays as proceeding not from the normal surface, but from the mirror, and (by an investigation perfectly analogous to that for the normals of a surface) to enquire as to the intersection of the ray by rays proceeding from consecutive points of the mirror; it would thus appear that there

are on the mirror two directions, such that proceeding along either of them to a consecutive point, the ray from the original point is intersected by the ray from the consecutive point, but that these directions are *not* in general at right angles, &c.

But in regard to such an investigation, the restriction introduced by the Hamiltonian theory is altogether unnecessary; there is no occasion to consider the rays which proceed from the several points of the mirror as being rays which are the normals of a surface, and the question is considered from the more general point of view as well by Malus in his "Théorie de la Double Refraction, &c.," Paris, 1810, as more recently by Kummer in the Memoir "Allgemeine Theorie der gradlinigen Strahlensysteme," *Crelle*, t. 57 (1860), pp. 189–230, viz. we have in Kummer a surface of any form whatever (defined according to the Gaussian theory, x, y, z given functions of the arbitrary parameters u, v), and from the several points thereof rays proceeding according to any law whatever, viz. the cosine-inclinations α, β, γ (or as Kummer writes them ξ, η, ζ) being given functions (such of course that $\alpha^2+\beta^2+\gamma^2=1$) of the same parameters u, v. It may be remarked: 1° that Kummer, while considering the simplifications of the general theory presenting themselves in the case where the rays are normals of the surface, does not specifically consider the case where, not being such normals, they are (as in the Hamiltonian theory) normals of a surface. 2° that some interesting investigations in regard to the shortest distances between consecutive rays, while naturally connecting themselves, with the normals of the surface, or with that of the rays which are normals of another surface, do not properly belong to the "Allgemeine Theorie" of a congruence, which is independent of the motion of rectangularity.

It has been already remarked that the system may be looked at in the two ways 1° and 2°, and it is in the former of these that the question is considered by Kummer; it is interesting to work out part of the theory in the latter of the two ways. Taking X, Y, Z as current coordinates, we have, for a line through the point (x, y, z), the equations

$$X, Y, Z = x+\alpha\rho,\ y+\beta\rho,\ z+\gamma\rho;$$

α, β, γ are functions of (x, y, z), satisfying identically the equation $\alpha^2+\beta^2+\gamma^2=1$ (and therefore the derived equations in regard to x, y, z respectively); and also satisfying the

equations

$$(\alpha\delta_x+\beta\delta_y+\gamma\delta_z)\,\alpha=0,\quad (\alpha\delta_x+\beta\delta_y+\gamma\delta_z)\,\beta=0,$$
$$(\alpha\delta_x+\beta\delta_y+\gamma\delta_z)\,\gamma=0.$$

It should be remarked that if these equations were not satisfied, then instead of a congruence there would be a complex, or triply infinite system of lines, viz. to the several points of space (x, y, z) there would correspond lines $X, Y, Z=x+\alpha\rho, y+\beta\rho, z+\gamma\rho$ as above, which lines would not reduce themselves to a doubly infinite system.

Suppose that the line through the point x, y, z is met by the line through a consecutive point $(x+dx, y+dy, z+dz)$; then, if X, Y, Z refer to the point of intersection of the two lines, we have

$$dx+\alpha d\rho+\rho d\alpha,\quad dy+\beta d\rho+\rho d\beta,\quad dz+\gamma d\rho+\rho d\gamma=0\,;$$

and consequently

$$\begin{vmatrix} dx, & d\alpha, & \alpha \\ dy, & d\beta, & \beta \\ dz, & d\gamma, & \gamma \end{vmatrix}=0$$

as a relation connecting the increments dx, dy, dz, in order that the lines may intersect, viz. this is a quadric relation $(*\between dx, dy, dz)^2=0$ between the increments. In the case of a complex this equation represents a cone (passing evidently through the line $dx:dy:dz=\alpha:\beta:\gamma$), but in the case of a congruence the cone must break up into a pair of planes intersecting in the line in question $dx:dy:dz=\alpha:\beta:\gamma$. To verify *à posteriori* that this is so, observe that the differential equations satisfied by α, β, γ give as above

$$\delta_y\gamma-\delta_z\beta,\quad \delta_z\alpha-\delta_x\gamma,\quad \delta_x\beta-\delta_y\alpha$$

proportional to α, β, γ, or say $=2k\alpha, 2k\beta, 2k\gamma$; and it hence follows that the differentials $\delta\alpha, \delta\beta, \delta\gamma$ can be expressed in the forms

$$\begin{aligned} d\alpha &= a\,dx+h\,dy+g\,dz+k\,(\beta\,dz-\gamma\,dy),\\ d\beta &= h\,dx+b\,dy+f\,dz+k\,(\gamma\,dx-\alpha\,dz),\\ d\gamma &= g\,dx+f\,dy+c\,dz+k\,(\alpha\,dy-\beta\,dx),\end{aligned}$$

where

$$\begin{aligned} 0&=a\alpha+h\beta+g\gamma,\\ 0&=h\alpha+b\beta+f\gamma,\\ 0&=g\alpha+f\beta+c\gamma.\end{aligned}$$

The equation
$$\begin{vmatrix} dx, & d\alpha, & \alpha \\ dy, & d\beta, & \beta \\ dz, & d\gamma, & \gamma \end{vmatrix} = 0$$
thus assumes the form
$$(a, b, c, f, g, h \between dx, dy, dz \between \gamma dy - \beta dz, \alpha dz - \gamma dx, \beta dx - \alpha dy)$$
$$+ k\{(\gamma dy - \beta dz)^2 + (\alpha dz - \gamma dx)^2 + (\beta dx - \alpha dy)^2\} = 0.$$
Write for shortness
$$\gamma dy - \beta dz, \quad \alpha dz - \gamma dx, \quad \beta dx - \alpha dy = \xi, \eta, \zeta;$$
then putting for a moment $adx + hdy + gdz = \Theta$, from this equation and $a\alpha + h\beta + g\gamma = 0$ we deduce
$$\alpha\Theta = -h(\beta dx - \alpha dy) + g(\alpha dz - \gamma dx),$$
that is $= -h\zeta + g\eta$; or, putting for Θ its value and forming the analogous equations, we have
$$\alpha(adx + hdy + gdz) = -h\zeta + g\eta,$$
$$\beta(hdx + bdy + fdz) = -f\xi + h\zeta,$$
$$\gamma(gdx + fdy + cdz) = -g\eta + f\xi,$$
and the quadric equation in (dx, dy, dz) thus becomes
$$\frac{\xi}{\alpha}(-h\zeta + g\eta) + \frac{\eta}{\beta}(-f\xi + h\eta) + \frac{\zeta}{\gamma}(-g\eta + f\zeta) + k(\xi^2 + \eta^2 + \zeta^2) = 0,$$
which, in virtue of the linear relation $\alpha\xi + \beta\eta + \gamma\zeta = 0$ connecting the (ξ, η, ζ), breaks up into a pair of planes, each passing through the line $\xi = 0$, $\eta = 0$, $\zeta = 0$ (that is $dx : dy : dz = \alpha : \beta : \gamma$).

We obtain at once as the condition that the two planes may be at right angles to each other $k = 0$; that is
$$\delta_y\gamma - \delta_z\beta, \quad \delta_z\alpha - \delta_x\gamma, \quad \delta_x\beta - \delta_y\alpha \text{ each } = 0;$$
or, what is the same thing, $\alpha dx + \beta dy + \gamma dz = dV$, a complete differential; and, as was seen, this is the condition in order that the lines may be the normals of a surface.

It thus appears that in a congruence each line is intersected by two consecutive lines which determine respectively two planes through the line; and further, that if for every line of the congruence, the two planes are at right angles to each other, then the consecutive lines are the normals of a surface.

ON A THEOREM OF PROF. KLEIN'S RELATING TO SYMMETRIC MATRICES.

By *Arthur Buchheim, M.A.*

THE theorem, that if a is a real symmetric matrix the roots of $|a-\lambda|=0$* are all real, is old and well known. Prof. Klein has given a generalisation of it in his paper on the most general linear transformation of line coordinates. Prof. Klein's theorem is as follows: Let a, b be two real symmetric matrices. Then, if, when we reduce bx^2† to a sum of real squares, the excess of the number of terms of one sign over the number of terms of the other sign is r, then the equation $|a-xb|=0$ has at least r real roots. I shall prove this by the method I have used to prove the particular case when $b=1$ in a former note in the *Messenger of Mathematics.* I use the notations employed in my paper on the theory of matrices and in my fourth note on the theory of screws in elliptic space in the *Proc. Lond. Math. Soc.* The proof can easily be translated into ordinary notation. I only consider the case in which a and b are not both vacuous.

Let b not be vacuous, then if e is a point such that $(a-\lambda b)e=0$, we have $b^{-1}ae=\lambda e$, so that e is a latent point of $b^{-1}a$; now let $Sxy=xKby$, $T^2x=Sxx$. Then it is easy to shew that if e, e' appertain to different latent roots of $a^{-1}b$, we have $See'=0$, and hence that if $\alpha+\iota\beta$, $\alpha-\iota\beta$ are latent points corresponding to imaginary roots $T^2\alpha+T^2\beta=0$.

Now let $\alpha_i\pm\iota\beta_i$ be the pairs of conjugate imaginary latent points, e_i the remaining latent points. Take these points of reference. Then, if

$$x=\Sigma\,(\xi_i\pm\iota\eta_i)\,(\alpha_i\pm\iota\beta_i)+\Sigma x_i e_i,$$

we easily see that

$$T^2x=2\Sigma\,(\xi_i^2-\eta_i^2)\,T^2\alpha_i+\Sigma x_i^2 T^2 e_i.$$

It is now obvious that the excess of the number of terms of one sign over the number of terms of the other sign is at most $n-2r$, where r is the number of imaginary roots of $|a-xb|=0$; but $n-2r$ is the number of real roots. Therefore the number of real roots is at least equal to the excess in question. This is Prof. Klein's theorem.

The Grammar School, Manchester,
June 4, 1887.

---

* I write $|a|$ for the content of the matrix a.

† If $b=\|b_{\iota\kappa}\|$ is a symmetric matrix, the quadric bx^2 is the quadric $\Sigma b_{\iota\kappa}x_\iota x_\kappa$.

A NEW METHOD FOR THE GRAPHICAL REPRESENTATION OF COMPLEX QUANTITIES.

By *J. Brill.*

1. As it is now more than eighty years since Argand introduced the well-known method for the graphical representation of complex quantities, and the principle of duality has long been recognized by geometers, it seems strange that it should have occurred to no one to apply the same idea to tangential coordinates; in other words to construct a diagram in which complex quantities should be represented by lines instead of points. It might be anticipated that the theorems obtained by this method would be none other than the polar reciprocals of theorems similarly obtained by the older method; however, since metrical properties offer considerable difficulty to reciprocation, it might be advisable to have a method which would yield the reciprocal properties directly. I have endeavoured to supply such a method in the following paper. It will be found that the theory is not an exact analogue of that in which complex quantities are represented by points. This is as might be expected, for it is well known that complete duality does not exist in the planimetry of homaloidal space.

2. Before proceeding to develope the theory we will prove a theorem which will be of use in the course of the investigation.

O is a fixed point, and a straight line is drawn through O meeting n fixed lines in the points $r_1, r_2, \ldots, r_n$. A point R is taken on the line through O, so that

$$\frac{m_1+m_2+\ldots+m_n}{OR}=\frac{m_1}{Or_1}+\frac{m_2}{Or_2}+\ldots+\frac{m_n}{Or_n}.$$

The locus of R will be a straight line.

To prove this take two rectangular axes through O, and let the equations of the given lines referred to these axes be

$$u_1x+v_1y-1=0,$$
$$u_2x+v_2y-1=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u_nx+v_ny-1=0.$$

Then, if θ be the angle made with the axis of x by the line through O, we have

$$\frac{1}{Or_1} = u_1 \cos\theta + v_1 \sin\theta, \quad \frac{1}{Or_2} = u_2 \cos\theta + v_2 \sin\theta,$$

....

....

$$\frac{1}{Or_n} = u_n \cos\theta + v_n \sin\theta.$$

Therefore the equation of the locus of R is

$$\frac{m_1 + m_2 + \ldots + m_n}{r} = m_1(u_1 \cos\theta + v_1 \sin\theta) + \ldots + m_n(u_n \cos\theta + v_n \sin\theta),$$

i.e.
$$x \frac{\Sigma mu}{\Sigma m} + y \frac{\Sigma mv}{\Sigma m} - 1 = 0.$$

We shall call this the mean line with respect to O of the n given lines for multiples m_1, m_2, ..., m_n.

3. Let the equation of a straight line be given in the form $ux + vy - 1 = 0$, where u and v are the reciprocals of the intercepts on the axes. We shall denote the position of this line by the expression $u + iv$, or if we desire to use a single symbol by w. This being premised, we see that the line at infinity will be represented by zero, and that any line through the origin will be represented by an infinite complex. Real quantities will represent lines parallel to the axis of y, and purely imaginary quantities will represent lines parallel to the axis of x. Further, any line parallel to the given one will be represented by a real multiple of the expression which denotes the given line.

4. Let there be two given straight lines whose equations are $ux + vy - 1 = 0$ and $u'x + v'y - 1 = 0$. Consider the line

$$m\,(ux + vy - 1) + n\,(u'x + v'y - 1) = 0,$$

or as it may be written

$$\frac{mu + nu'}{m + n} x + \frac{mv + nv'}{m + n} y - 1 = 0.$$

Let this be equivalent to $Ux + Vy - 1 = 0$, then we have

$$(m + n)\,U = mu + nu' \quad \text{and} \quad (m + n)\,V = mv + nv'.$$

Hence, if $W=U+iV$, $w=u+iv$, and $w'=u'+iv'$, we have

$$(m+n)\,W=mw+nw'.$$

The line W is what we have called the mean line of the lines w and w' for multiples m and n.

If we make $m=n$, then we have $2W=w+w'$. In this case W will coincide with the harmonic polar of the origin with respect to the lines w and w'. Thus to obtain the line which is the sum of two given lines, we have to draw the harmonic polar of the origin with respect to them, and then to draw a line parallel to this and at one-half the distance from the origin. Thus the idea of the mean line furnishes us with an interpretation of the addition of lines, just as the idea of the mean centre furnishes us with an interpretation of the addition of points. The actual position of the mean line, however, depends upon the position of the origin, so that the analogy is not quite complete.

If we write $m+n=0$, then W becomes a line through the origin, and all that we can say is that $w-w'$ is some line parallel to this. It will therefore be necessary to investigate the case of $w-w'$ separately. The equation of this line is

$$(u-u')\,x+(v-v')\,y-1=0.$$

This is obviously parallel to

$$(u-u')\,x+(v-v')\,y=0,$$

the line joining the origin to the intersection of w and w'. In fact this latter line coincides with W. Further, the equation of $w-w'$ may be written

$$ux+vy-1-(u'x+v'y)=0,$$

which shews that it passes through the intersection of w with a line drawn through the origin parallel to w'. Thus, to construct the line $w-w'$, we draw through the origin a line parallel to w', and through the intersection of this with w we draw a line parallel to that joining the origin to the intersection of w and w'.

If W be the mean of the lines $w_1, w_2, \ldots, w_n$ for multiples $m_1, m_2, \ldots, m_n$, then, by proceeding as in the early part of this article, we should obtain

$$(m_1+m_2+\ldots+m_n)\,W=m_1w_1+m_2w_2+\ldots+m_nw_n.$$

5. Let the straight line $ux+vy-1=0$ cut the axis of x at the point A, and that of y at the point B; and let $OA=a$ and $OB=b$. From O draw OL perpendicular to AB. Let

$OL=p$, and let θ be the angle which OD makes with the axis of x. Then we have

$$u=\frac{1}{a}=\frac{\cos\theta}{p}, \text{ and } v=\frac{1}{b}=\frac{\sin\theta}{p}.$$

Therefore

$$w=u+iv=\frac{1}{p}(\cos\theta+i\sin\theta)=\frac{e^{i\theta}}{p}.$$

This formula will enable us to discuss the question of the multiplication of two lines. For let $w/c=w_1w_2$, where w, w_1, w_2 denote lines and c denotes some length. Substituting for w, w_1 and w_2 their values as given by the above formula we obtain

$$\frac{\cos\theta+i\sin\theta}{cp}=\frac{\cos\theta_1+i\sin\theta_1}{p_1}\,\frac{\cos\theta_2+i\sin\theta_2}{p_2}$$
$$=\frac{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)}{p_1p_2}.$$

Hence, we have

$$\frac{\cos\theta}{cp}=\frac{\cos(\theta_1+\theta_2)}{p_1p_2} \text{ and } \frac{\sin\theta}{cp}=\frac{\sin(\theta_1+\theta_2)}{p_1p_2},$$

from which it follows that

$$cp=p_1p_2 \text{ and } \theta=\theta_1+\theta_2.$$

In a similar manner it can be proved that if $w^2=w_1w_2$ then

$$p^2=p_1p_2 \text{ and } 2\theta=\theta_1+\theta_2.$$

6. As a first example of the utility of the method we will investigate the analogue of the well-known proof by Argand's method of Euc. I. 47. Taking the figure of the last article produce BO to B', making $OB'=OB$. Then the line AB will be represented by $1/a+i/b$, and the line AB' by $1/a-i/b$. Further, from the last article we have

$$\frac{1}{a}+\frac{i}{b}=\frac{e^{i\theta}}{p}.$$

Also, the perpendicular from O on AB' is equal in length to that from O on AB, and the two perpendiculars make equal angles with the axis of x on opposite sides of it; therefore

$$\frac{1}{a}-\frac{i}{b}=\frac{e^{-i\theta}}{p}.$$

Multiplying together these two equations we obtain

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2},$$

which expresses a well-known property of the right-angled triangle.

7. We will now proceed to develope some metrical properties connected with the theory of the mean line.

Let w be the mean of the lines w_1 and w_2 for multiples m and n, then we have

$$(m+n)\,w=mw_1+nw_2.$$

This is equivalent to the two metrical relations

$$(m+n)\frac{\cos\theta}{p}=m\frac{\cos\theta_1}{p_1}+n\frac{\cos\theta_2}{p_2}$$

and $$(m+n)\frac{\sin\theta}{p}=m\frac{\sin\theta_1}{p_1}+n\frac{\sin\theta_2}{p_2}.$$

From these we easily deduce the two following:

$$\frac{(m+n)^2}{p^2}=\frac{m^2}{p_1^{\,2}}+\frac{n^2}{p_2^{\,2}}+2\frac{mn}{p_1p_2}\cos(\theta_1-\theta_2)$$

and $$\frac{m+n}{p}=\frac{m}{p_1}\cos(\theta-\theta_1)+\frac{n}{p_2}\cos(\theta-\theta_2).$$

Let AX, AB, AC be the lines respectively denoted by w, w_1, w_2. From O draw OL, OM, ON respectively perpendicular to AX, AB, AC. Then, if O lie within the angle BAC or the vertically opposite angle, we have

$$\frac{(m+n)^2}{OL^2}=\frac{m^2}{OM^2}+\frac{n^2}{ON^2}-2\frac{mn}{OM.ON}\cos BAC;$$

but, if O lie without both these angles, we have

$$\frac{(m+n)^2}{OL^2}=\frac{m^2}{OM^2}+\frac{n^2}{ON^2}+2\frac{mn}{OM.ON}\cos BAC.$$

Through L draw LH and LK respectively perpendicular to OM and ON. The second of the above relations becomes

$$m\frac{OH}{OM}+n\frac{OK}{ON}=m+n.$$

Both these properties admit of generalization, but we will only trouble ourselves with the second. Let

$$(m_1+m_2+\ldots+m_n)\,w = m_1w_1+m_2w_2+\ldots+m_nw_n.$$

This is equivalent to the two metrical relations

$$(m_1+m_2+\ldots+m_n)\frac{\cos\theta}{p} = m_1\frac{\cos\theta_1}{p_1}+m_2\frac{\cos\theta_2}{p_2}+\ldots+m_n\frac{\cos\theta_n}{p_n}$$

and

$$(m_1+m_2+\ldots+m_n)\frac{\sin\theta}{p} = m_1\frac{\sin\theta_1}{p_1}+m_2\frac{\sin\theta_2}{p_2}+\ldots+m_n\frac{\sin\theta_n}{p_n}.$$

From these we deduce

$$\frac{m_1+m_2+\ldots+m_n}{p} = \frac{m_1}{p_1}\cos(\theta-\theta_1)+\frac{m_2}{p_2}\cos(\theta-\theta_2)+\ldots+\frac{m_n}{p_n}\cos(\theta-\theta_n).$$

Let OL, OM_1, OM_2, ..., OM_n be the perpendiculars from O on w, w_1, w_2, ..., w_n. From L draw LN_1, LN_2, ..., LN_n respectively perpendicular to OM_1, OM_2, ..., OM_n. The above relation becomes

$$m_1\frac{ON_1}{OM_1}+m_2\frac{ON_2}{OM_2}+\ldots+m_n\frac{ON_n}{OM_n} = m_1+m_2+\ldots+m_n,$$

or as it may be written

$$m_1\frac{M_1N_1}{OM_1}+m_2\frac{M_2N_2}{OM_2}+\ldots+m_n\frac{M_nN_n}{OM_n} = 0.$$

8. The relation $(m+n)\,w = mw_1+nw_2$ may be written in the form

$$(m+n)\frac{e^{i\theta}}{p} = m\frac{e^{i\theta_1}}{p_1}+n\frac{e^{i\theta_2}}{p_2}.$$

Hence, if r be a positive integer,

$$(m+n)^r\frac{e^{ir\theta}}{p^r} = m^r\frac{e^{ir\theta_1}}{p_1^r}+rm^{r-1}n\frac{e^{i\{(r-1)\theta_1+\theta_2\}}}{p_1^{r-1}p_2}$$
$$+\frac{r(r-1)}{2!}m^{r-2}n^2\frac{e^{i\{(r-2)\theta_1+2\theta_2\}}}{p_1^{r-2}p_2^2}+\ldots+n^r\frac{e^{ir\theta_2}}{p_2^r}.$$

If we split this into its real and imaginary parts, multiply the first by $\cos r\theta$ and the second by $\sin r\theta$ and add, we have

$$\frac{(m+n)^r}{p^r} = \frac{m^r}{p_1^r}\cos r(\theta_1-\theta) + r\frac{m^{r-1}n}{p_1^{r-1}p_2}\cos\{(r-1)(\theta_1-\theta)+\theta_2-\theta\}$$
$$+\ldots+\frac{n^r}{p_2^r}\cos r(\theta_2-\theta).$$

It would be troublesome to enumerate all the cases of this that might arise, so we will content ourselves with one particular case. Let O lie without both the angle BAC and the vertically opposite angle, and let m and n be both of the same sign so that AX lies between AB and AC. The above property becomes

$$\frac{(m+n)^r}{OL^r} = \frac{m^r}{OM^r}\cos\{r.BAX\}$$
$$+r\frac{m^{r-1}n}{OM^{r-1}.ON}\cos\{(r-1)BAX-CAX\}$$
$$+\frac{r(r-1)}{2!}\cdot\frac{m^{r-2}n^2}{OM^{r-2}.ON^2}\cos\{(r-2)BAX-2.CAX\}$$
$$+\ldots+\frac{n^r}{ON^r}\cos\{r.CAX\}.$$

This is the generalized analogue of a property of the triangle published by me in the *Educational Times* for October 1885 (No. 8290).

9. Let w_2 and w_3 denote the lines AB and AC, the origin O lying within the angle BAC. Draw AX the harmonic polar of O with the respect to the lines AB and AC. On the opposite side of O from AX, and at a distance from O equal to one-half of that of AX from O, draw a line BC. Then, if this line be denoted by w_1, we have $w_1+w_2+w_3=0$. This is equivalent to the two metrical relations

$$\frac{\cos\theta_1}{p_1}+\frac{\cos\theta_2}{p_2}+\frac{\cos\theta_3}{p_3}=0$$

and

$$\frac{\sin\theta_1}{p_1}+\frac{\sin\theta_2}{p_2}+\frac{\sin\theta_3}{p_3}=0,$$

from which we deduce

$$\frac{1/p_1}{\sin(\theta_2-\theta_3)}=\frac{1/p_2}{\sin(\theta_3-\theta_1)}=\frac{1/p_3}{\sin(\theta_1-\theta_2)}.$$

If OL, OM, ON be the perpendiculars from O on BC, CA, AB respectively, this relation becomes

$$OL\sin BAC=OM\sin CBA=ON\sin ACB,$$

which shews that O is the centroid of the triangle ABC.

Further, since $w_1+w_2+w_3=0$, we have

$$w_1^3+w_2^3+w_3^3=3w_1w_2w_3.$$

This is equivalent to the two metrical relations

$$\frac{\cos 3\theta_1}{p_1^3}+\frac{\cos 3\theta_2}{p_2^3}+\frac{\cos 3\theta_3}{p_3^3}=3\frac{\cos(\theta_1+\theta_2+\theta_3)}{p_1p_2p_3}$$

and
$$\frac{\sin 3\theta_1}{p_1^3}+\frac{\sin 3\theta_2}{p_2^3}+\frac{\sin 3\theta_3}{p_3^3}=3\frac{\sin(\theta_1+\theta_2+\theta_3)}{p_1p_2p_3},$$

from which we deduce

$$\frac{\cos(2\theta_1-\theta_2-\theta_3)}{p_1^3}+\frac{\cos(2\theta_2-\theta_3-\theta_1)}{p_2^3}+\frac{\cos(2\theta_3-\theta_1-\theta_2)}{p_3^3}=\frac{3}{p_1p_2p_3}.$$

Hence, we have the following theorem:

If O be the centroid of the triangle ABC, and OL, OM, ON be the respective perpendiculars from O on BC, CA, AB, then

$$\frac{\cos(B-C)}{OL^3}+\frac{\cos(C-A)}{OM^3}+\frac{\cos(A-B)}{ON^3}=\frac{3}{OL.OM.ON}.$$

10. Let $w_1+w_2+w_3$ and $s_1+s_2+s_3=0$, then we have

$$s_2w_3-s_3w_2=s_3w_1-s_1w_3=s_1w_2-s_2w_1,$$

or as it may be written

$$\frac{s_2}{w_2}-\frac{s_3}{w_3}:\frac{s_3}{w_3}-\frac{s_1}{w_1}:\frac{s_1}{w_1}-\frac{s_2}{w_2}::w_1:w_2:w_3.$$

These six complexes being proportional, it follows that their moduli must be proportional. Let p_1, p_2, p_3, q_1, q_2, q_3 be the perpendiculars from the origin on w_1, w_2, w_3, s_1, s_2, s_3 respectively, and let θ_1, θ_2, θ_3, ϕ_1, ϕ_2, ϕ_3 be the respective angles that these perpendiculars make with the axis of x.

Then the expression $s_2/w_2 - s_3/w_3$ is equal to

$$\frac{p_2}{q_2}\cos(\phi_2-\theta_2)-\frac{p_3}{q_3}\cos(\phi_3-\theta_3)+i\left\{\frac{p_2}{q_2}\sin(\phi_2-\theta_2)-\frac{p_3}{q_3}\sin(\phi_3-\theta_3)\right\},$$

and therefore its modulus is

$$\left(\frac{p_2}{q_2}\right)^2+\left(\frac{p_3}{q_3}\right)^2-2\frac{p_2p_3}{q_2q_3}\cos\{\phi_2-\phi_3-(\theta_2-\theta_3)\}.$$

Hence, we have the following theorem:

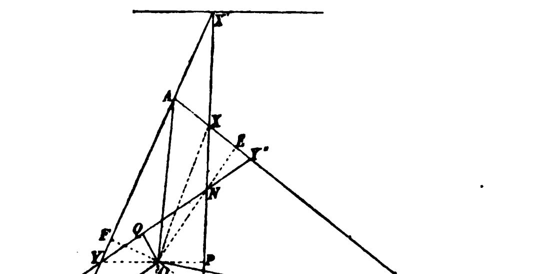
Let ABC and DEF be two triangles having a common centroid O, and let OL, OM, ON, OP, OQ, OR be the respective perpendiculars from O on BC, CA, AB, EF, FD DE; then we have

$$\left(\frac{OM}{OQ}\right)^2+\left(\frac{ON}{OR}\right)^2-2\,\frac{OM.ON}{OQ.OR}\cos(A-D)$$

$$:\left(\frac{ON}{OR}\right)^2+\left(\frac{OL}{OP}\right)^2-2\,\frac{ON.OL}{OR.OP}\cos(B-E)$$

$$:\left(\frac{OL}{OP}\right)^2+\left(\frac{OM}{OQ}\right)^2-2\,\frac{OL.OM}{OP.OQ}\cos(C-F)::\frac{1}{OL^2}:\frac{1}{OM^2}:\frac{1}{ON^2}.$$

11. Let ABC be a triangle. Take a point O within the triangle, join OA, OB, OC, and draw OD, OE, OF respectively perpendicular to BC, CA, AB.



Through O draw OX parallel to AB to meet CA in X, and through X draw MN parallel to OA. Through O draw OY parallel to BC to meet AB in Y, and through Y draw NL parallel to OB. Through O draw OZ parallel to CA to meet BC in Z, and

through Z draw LM parallel to OC. Draw OP, OQ, OR respectively perpendicular to MN, NL, LM.

If the lines BC, CA, AB be denoted by w_1, w_2, w_3 then MN, NL, LM will be denoted by $w_2 - w_3$, $w_3 - w_1$, $w_1 - w_2$. We shall, therefore, be able to deduce a theorem from our figure by means of the identity

$$w_1(w_2 - w_3) + w_2(w_3 - w_1) + w_3(w_1 - w_2) = 0.$$

If $p_1, p_2, p_3, q_1, q_2, q_3$ stand for OD, OE, OF, OP, OQ, OR, and $\theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3$ be the angles made by these lines with the axis of x; then, proceeding as in the early part of Art. 9, we shall arrive at the result

$$p_1q_1 \sin(\theta_2 - \theta_3 + \phi_2 - \phi_3) = p_2q_2 \sin(\theta_3 - \theta_1 + \phi_3 - \phi_1)$$
$$= p_3q_3 \sin(\theta_1 - \theta_2 + \phi_1 - \phi_2),$$

i.e. $OD.OP \sin(BAC + MLN) = OE.OQ \sin(CBA + NML)$
$$= OF.OR \sin(ACB + LNM).$$

12. The same figure also furnishes us with interpretations of the formulæ

$$w_1^2(w_2 - w_3) + w_2^2(w_3 - w_1) + w_3^2(w_1 - w_2)$$
$$+ (w_2 - w_3)(w_3 - w_1)(w_1 - w_2) = 0$$

and

$$w_2w_3(w_2 - w_3) + w_3w_1(w_3 - w_1) + w_1w_2(w_1 - w_2)$$
$$+ (w_2 - w_3)(w_3 - w_1)(w_1 - w_2) = 0.$$

The first of these is equivalent to the two metrical relations

$$\frac{\cos(2\theta_1 + \phi_1)}{p_1^2q_1} + \frac{\cos(2\theta_2 + \phi_2)}{p_2^2q_2} + \frac{\cos(2\theta_3 + \phi_3)}{p_3^2q_3}$$
$$+ \frac{\cos(\phi_1 + \phi_2 + \phi_3)}{q_1q_2q_3} = 0$$

and

$$\frac{\sin(2\theta_1 + \phi_1)}{p_1^2q_1} + \frac{\sin(2\theta_2 + \phi_2)}{p_2^2q_2} + \frac{\sin(2\theta_3 + \phi_3)}{p_3^2q_3}$$
$$+ \frac{\sin(\phi_1 + \phi_2 + \phi_3)}{q_1q_2q_3} = 0,$$

from which we deduce

$$\frac{\cos(2\theta_1 - \phi_2 - \phi_3)}{p_1^2q_1} + \frac{\cos(2\theta_2 - \phi_3 - \phi_1)}{p_2^2q_2} + \frac{\cos(2\theta_3 - \phi_1 - \phi_2)}{p_3^2q_3}$$
$$+ \frac{1}{q_1q_2q_3} = 0.$$

Let MN, NL, LM meet AB, BC, CA respectively in X', Y', Z'. Then this relation becomes

$$\frac{\cos(LY'Z-LZY')}{OD^2.OP}+\frac{\cos(MZ'X-MXZ')}{OE^2.OQ}$$
$$\frac{\cos(NX'Y-NYX')}{OF^2.OR}=\frac{1}{OP.OQ.OR}.$$

Similarly we may deduce from the second formula the relation

$$\frac{\cos(AY''Y-AZ''Z)}{OE.OF.OP}+\frac{\cos(BZ''Z-BX''X)}{OF.OD.OQ}$$
$$+\frac{\cos(CX''X-CY''Y)}{OD.OE.OR}=\frac{1}{OP.OQ.OR},$$

where X'', Y'', Z'' are the respective intersections of MN, NL, LM with BC, CA, AB.

13. We could obtain analogues of all the properties of rectilinear figures obtained in my paper on Argand's method.* We will, however, only consider one more instance, viz. the interpretation of the formula

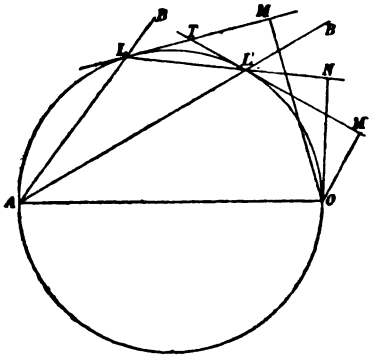
$$w_1^2(w_2+w_3)+w_2^2(w_3+w_1)+w_3^2(w_1+w_2)$$
$$=w_2w_3(w_2+w_3)+w_3w_1(w_3+w_1)+w_1w_2(w_1+w_2).$$

Let ABC be a triangle and O a point within it. Let MN be the harmonic polar of O with respect to CA and AB, NL that of O with respect to AB and BC, and LM that of O with respect to BC and CA. Draw OD, OE, OF, OP, OQ, OR respectively perpendicular to BC, CA, AB, MN, NL, LM. Then we have

$$\frac{\cos(LBC-LCB)}{OD^2.OP}+\frac{\cos(MCA-MAC)}{OE^2.OQ}+\frac{\cos(NAB-NBA)}{OF^2.OR}$$
$$=\frac{\cos(MCA-NBA)}{OE.OF.OP}+\frac{\cos(NAB-LCB)}{OF.OD.OQ}+\frac{\cos(LBC-MAC)}{OD.OE.OR}.$$

14. Let O be the origin and A a fixed point distant $2a$ from O. On OA as diameter describe a circle, and through A draw a line AB cutting the circle in L. Draw a tangent LM to the circle, join OL, and draw OM perpendicular

* *Messenger of Mathematics*, vol. XVI., pp. 8–20.



to LM. Then we have $OM.OA = OL^2$, and the angle MOA is bisected by OL. Thus, if w represent the line AB, then LM is represented by $2aw^2$.

Through A draw another line AB' meeting the circle in L', and join OL'. Draw $L'M'$ the tangent at L', meeting LM in T, and draw OM' perpendicular to $L'M'$. Join LL' and draw ON perpendicular to it. Join $M'N$ and NM. Then we see that the angle

$$M'NO = M'L'O = L'AO = L'LO = NMO.$$

Also the angle

$$M'ON = M'L'N = TL'L = TLL' = MON.$$

Thus the triangles $M'ON$ and MON are similar, and therefore $OM.OM' = ON^2$. Now we have

$$OA.OM = OL^2 \quad \text{and} \quad OA.OM' = OL'^2;$$

consequently

$$OL^2.OL'^2 = OA^2.OM.OM' = OA^2.ON^2,$$

and therefore $$OL.OL' = OA.ON.$$

Further,

$$2.AON = AOM + AOM' = 2(AOL + AOL');$$

that is, $$AON = AOL + AOL'.$$

Hence we see that if AB be represented by w and AB' by w', then LL' is represented by $2aww'$.

Thus we see that the circle can be utilized to obtain interpretations of formulæ containing squares and products. The method of interpretation may however be made more general than that given above. We have taken A, the point diametrically opposite the origin, as our fixed point, and the diameter OA as our initial direction. Suppose that we now take L for our fixed point, and OM the perpendicular from O

on the tangent at L for our initial direction. We have $ON^2 = OM.OM'$, and the angle $MOM' = 2.MON$. Thus, if $OM = 2c$, and LL' be represented by w_1, then $L'M'$ is represented by $2cw_1^2$.

Draw another line LL'' through L, and draw ON' perpendicular to LL''. Let $L''M''$ be the tangent at L'', and OM'' the perpendicular on it from the origin. Join $L'L''$, and draw OK perpendicular to it. Then we have

$$ON^2 = OM.OM' \quad \text{and} \quad ON'^2 = OM.OM'',$$

and therefore

$$ON^2.ON'^2 = OM^2.OM'.OM'' = OM^2.OK^2;$$

that is, $$ON.ON' = OM.OK.$$

Further, OK bisects the angle $M'OM''$, and therefore

$$2.MOK = MOM' + MOM'' = 2(MON + MON');$$

that is, $$MOK = MON + MON'.$$

Thus, if LL' be represented by w_1, and LL'' by w_2, then $L'L''$ is represented by $2cw_1w_2$.

15. As a first example of the method of interpretation developed in the last article we will take the formula for the product of two polynomials. This will be easily seen to yield the following theorem:

O and L are two fixed points on the circumference of a circle. A series of m fixed points is taken on the circumference of the same circle, and a second series containing n points is also taken. Let LM be the mean line for equal multiples with respect to O of the lines joining L to the m points, and let LN be the mean line for equal multiples with respect to O of the lines joining L to the n points. Let LM and LN meet the circle in M and N respectively; then MN is the mean line for equal multiples with respect to O of the mn lines that can be obtained by joining any one of the m points to any one of the n points.

16. We will next take the formula for the square of a polynomial, viz.

$$(w_1 + w_2 + \ldots + w_n)^2 = \Sigma w^2 + 2\Sigma ww'.$$

As before, let O be the origin and L another fixed point on the circumference of the circle. Take n other fixed points on the circle, and let the lines joining L to these points be

denoted by $w_1, w_2, \ldots, w_n$. Then, if W be the mean line for equal multiples with respect to O of these lines, we have $nW = w_1 + w_2 + \ldots + w_n$. Let PX be the mean line for equal multiples with respect to O of the tangents at the n points, and let QY be that of the $\frac{1}{2}n(n-1)$ lines that can be obtained by joining the n points two and two. Then, if PX and QY be denoted by W_1 and W_2, we have

$$nW_1 = \Sigma w^2 \quad \text{and } n(n-1)\, W_2 = 2\Sigma ww',$$

and therefore $\qquad nW^2 = W_1 + (n-1)\, W_2.$

Thus, if W cut the circle in M, the tangent at M is the mean line with respect to O of PX and QY for the respective multiples $1, n-1$.

Further, it is evident that if we make L moveable, M will remain fixed, and the envelope of LM will be the fixed point M. Thus we are led to the following theorem:

Let O be a fixed and L a moveable point on the circumference of a circle, and let n fixed points be taken on the circle. Let PX be the mean line for equal multiples with respect to O of the tangents at the n points, and QY that of the secants that can be obtained by joining the n points. Then the mean line for equal multiples with respect to O of the lines joining L to the n points passes through a fixed point M on the circumference of the circle, and the tangent at M is the mean line with respect to O of PX and QY for the respective multiples $1, n-1$.

17. As a final example we will give the interpretation of the formula

$$w_1(w'_2 + w'_3) + w_2(w'_3 + w'_1) + w_3(w'_1 + w'_2)$$
$$= w'_1(w_2 + w_3) + w'_2(w_3 + w_1) + w'_3(w_1 + w_2).$$

Let O, L, A, B, C, D, E, F be eight fixed points on the circumference of a circle. Let LP be the harmonic polar of O with respect to LB and LC, LQ that of O with respect to LC and LA, LR that with respect to LA and LB, LX that with respect to LE and LF, LY that with respect to LF and LD, and LZ that with respect to LD and LE, the six points P, Q, R, X, Y, Z lying on the circumference of the circle. Then the mean line for equal multiples with respect to O of the lines AX, BY, CZ coincides with that of the lines DP, EQ, FR.

NOTE ON THE TWO RELATIONS CONNECTING THE DISTANCES OF FOUR POINTS ON A CIRCLE.

By Prof. *Cayley*.

CONSIDER a quadrilateral $BACD$ inscribed in a circle; and let the sides BA, AC, CD, DB and diagonals BC and AD be $= c, b, h, g, a, -f$ respectively; f is for convenience taken negative, so that the equation connecting the sides and diagonals may be

$$\Delta, \quad = af + bg + ch, \quad = 0.$$

We have between the sides and diagonals another relation

$$V, \quad = abc + agh + bhf + cfg, \quad = 0,$$

as is easily proved geometrically; in fact, recollecting that the opposite angles are supplementary to each other, the double area of the quadrilateral is $= (bc + gh) \sin A$, and it is also $= (bh + cg) \sin B$; that is, we have

$$(bc + gd) \sin A - (bh + cg) \sin B = 0.$$

But from the triangles BAD and BAC, in which the angles D, C are equal to each other, we have

$$\frac{c}{\sin D} = -\frac{f}{\sin B}, \quad \frac{c}{\sin C} = \frac{a}{\sin A};$$

that is
$$f \sin A + a \sin B = 0;$$

and thence the required relation

$$a(bc + gh) + f(bh + cg) = 0.$$

The distances of the four points on the circle are thus connected by the two equations $\Delta = 0$, $V = 0$. Considering a, b, c, f, g, h as the distances from each other of any four points in the plane, we have between them the relation

$$\begin{aligned} \Omega, \quad = \quad & a^2f^2(-a^2 - f^2 + b^2 + g^2 + c^2 + h^2) \\ & + b^2g^2(\ \ a^2 + f^2 - b^2 - g^2 + c^2 + h^2) \\ & + c^2h^2(\ \ a^2 + f^2 + b^2 + g^2 - c^2 - h^2) \\ & - a^2b^2c^2 - a^2g^2h^2 - b^2h^2f^2 - c^2f^2g^2, \quad = 0; \end{aligned}$$

and it is clear that this equation should be a consequence of the equations $\Delta = 0$, $V = 0$. To verify this, forming the sum $\Omega + V^2$, we have

$$\begin{aligned} \Omega + V^2 = \quad & (a^2 + f^2)(-a^2f^2 + b^2g^2 + c^2h^2 + 2bgch) \\ & + (b^2 + g^2)(-b^2g^2 + c^2h^2 + a^2f^2 + 2chaf) \\ & + (c^2 + h^2)(-c^2h^2 + a^2f^2 + b^2g^2 + 2afbg); \end{aligned}$$

viz. this is

$$\begin{aligned} = \ & (a^2+f^2)\{-a^2f^2+(\Delta-af)^2\} \\ & + (b^2+g^2)\{-b^2g^2+(\Delta-bg)^2\} \\ & + (c^2+h^2)\{-c^2h^2+)\Delta-ch)^2\}; \end{aligned}$$

or, since

$$-a^2f^2+(\Delta-af)^2=\Delta(\Delta-2af)=\Delta(-af+bg+ch), \text{ \&c.,}$$

this is

$$\begin{aligned} \Omega+V^2=\Delta[\ & (a^2+f^2)(-af+bg+ch) \\ & +(b^2+g^2)(\ \ af-bg+ch) \\ & +(c^2+h^2)(\ \ af+bg-ch)], \end{aligned}$$

which proves the theorem.

It may be remarked that the equation $V=0$ may be written

$$a(bc+gh)+f(bh+cg)=0;$$

viz., multiplying by a, and for af writing its value, $=-(bg+ch)$ from the equation $\Delta=0$, this gives

$$-a^2(bc+gh)+(bg+ch)(bh+cg)=0;$$

that is $\qquad bc(g^2+h^2-a^2)+gh(b^2+c^2-a^2)=0,$

which expresses that the angles A, D are supplementary to each other; and, similarly, by the elimination of any other of the six quantities from the equations $\Delta=0$, $V=0$, we have five other like equations.

NOTE ON THE ANHARMONIC RATIO EQUATION.

By Prof. *Cayley.*

Given any four quantities α, β, γ, δ, if θ be one of the values of the anharmonic ratio, the other values are

$$\frac{1}{\theta},\quad -(1+\theta),\quad -\frac{1}{1+\theta},\quad -\frac{\theta}{1+\theta},\quad -\frac{1+\theta}{\theta};$$

and hence the equation having these six roots is

$$(x-\theta)\left(x-\frac{1}{\theta}\right)(x+1+\theta)$$
$$\left(x+\frac{1}{1+\theta}\right)\left(x+\frac{1}{1+\theta}\right)\left(x+\frac{1+\theta}{\theta}\right)=0;$$

or, multiplying out, the equation, as is well-known, takes the form

$$(x^2+x+1)^3 - \frac{(\theta^2+\theta+1)^3}{\theta^2(\theta+1)^2}x^2(x+1)^2 = 0.$$

But to effect the multiplication in the easiest manner we may proceed as follows: writing

$$a,\ b,\ c = (\alpha-\delta)(\beta-\gamma),\quad (\beta-\delta)(\gamma-\alpha),\quad (\gamma-\delta)(\alpha-\beta),$$

so that $a+b+c=0$, the equation is

$$\left(x-\frac{b}{c}\right)\left(x-\frac{c}{b}\right)\left(x-\frac{c}{a}\right)\left(x-\frac{a}{c}\right)\left(x-\frac{a}{b}\right)\left(x-\frac{b}{a}\right)=0.$$

The product of the first pair of factors is

$$x^2+1-\left(\frac{b}{c}+\frac{c}{b}\right)x,\quad =(x+1)^2-\frac{a^2}{bc}x;$$

thus the equation is

$$\left\{(x+1)^2-\frac{a^2}{bc}x\right\}\left\{(x+1)^2-\frac{b^2}{ca}x\right\}\left\{(x+1)^2-\frac{c^2}{ab}x\right\}=0;$$

that is,

$$(x+1)^6-\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)x(x+1)^4$$
$$+\left(\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}\right)x^2(x+1)^2-1=0;$$

and recollecting that $a+b+c=0$, and writing $q=bc+ca+ab$, $r=abc$, the equation becomes

$$(x+1)^6-3(x+1)^4x+\left(3+\frac{q^3}{r^2}\right)(x+1)^2x^2-1=0;$$

that is

$$(x^2+x+1)^3+\frac{q^3}{r^2}(x+1)^2x^2=0.$$

But, writing $\theta=\dfrac{b}{a}$, we have

$$(\theta^2+\theta+1)^3+\frac{q^3}{r^2}(\theta+1)^2\theta^2=0;$$

or finally

$$(x^2+x+1)^3-\frac{(\theta^2+\theta+1)^3}{\theta^2(\theta+1)^2}x^2(x+1)^2=0,$$

the required result.

CONTENTS.

The following papers have been received:

Major Allan Cunningham, "On the depression of differential equations."

Mr. R. Tucker, "The cosine orthocentres and a cubic through them."

Mr. G. G. Morrice, "Note on the multiplication of nonions."

Mr. C. Chree, "Vortices in a compressible fluid."

Articles for insertion will be received by the Editor, or by Messrs. Metcalfe and Son, Printing Office, Trinity Street, Cambridge.

No. CXCIX.] NEW SERIES. [November, 1887.

THE MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XVII.—NO. 7.

MACMILLAN AND CO.

London and Cambridge.

1887.

Price—One Shilling.

METCALFE AND SON. CAMBRIDGE.

THE "COSINE" ORTHOCENTRES OF A TRIANGLE AND A CUBIC THROUGH THEM.

By *R. Tucker, M.A.*

ABC is a triangle, of which AL, BM, CN are the altitudes co-intersecting in the orthocentre H.

LF, LE' are drawn parallel to AC, AB;
MD, MF' .... AB, BC;
and NE, ND' .... BC, CA;
the lines AD, BE, CF cointersect in σ_1,
and the lines AD', BE', CF' cointersect in σ_2.

The equations (in trilinear coordinates) to AD, BE, CF are respectively

$b\beta\cos A = a\gamma\cos C$, $c\gamma\cos B = b\alpha\cos A$, $a\alpha\cos C = c\beta\cos B$...(i),

and to AD', BE', CF' are

$a\beta\cos B = c\gamma\cos A$, $b\gamma\cos C = a\alpha\cos B$, $c\alpha\cos A = b\beta\cos C$...(ii),

hence σ_1, σ_2 are given by the equations

$$\left.\begin{aligned}\frac{\alpha}{c\cos B} = \frac{\beta}{a\cos C} = \frac{\gamma}{b\cos A} = \frac{4\Delta}{a^2+b^2+c^2} = \tan\omega \\ \frac{\alpha}{b\cos C} = \frac{\beta}{c\cos A} = \frac{\gamma}{a\cos B} = \tan\omega\end{aligned}\right\}\text{(iii).}$$

From (iii) it is at once seen that the Lemoine point (K) is the mid-point of the join of σ_1, σ_2 .... (iv).

If now l_1, l_2; m_1, m_2; n_1, n_2 are the projections of σ_1, σ_2 on the altitudes, then Bl_1, Cm_1, An_1 cointersect in Ω, and Cl_2, Am_2, Bn_2 in Ω' ....(v).

Again Cl_1, Am_1, Bn_1; Bl_2, Cm_2, An_2 cointersect respectively in

$\epsilon_1\,(a\cot B,\ b\cot C,\ c\cot A)$

and $\epsilon_2\,(a\cot C,\ b\cot A,\ c\cot B)$ ....(vi).

The mid-point (η) of the join of ϵ_1, ϵ_2 is

$a^3 : b^3 : c^3$........................(vii),

and lies on the circum-Brocard axis; this axis therefore bisects $\sigma_1\sigma_2$, $\Omega\Omega'$ and $\epsilon_1\epsilon_2$.

We readily obtain the following results:

$$\left.\begin{aligned} AE &= b^2 \cos A/c, & CE &= ab\cos B/c,\\ AE' &= bc\cos B/a, & CE' &= b^2\cos C/a,\\ CD &= a^2\cos C/b, & BD &= ac\cos A/b,\\ CD' &= ab\cos A/c, & BD' &= a^2\cos B/c,\\ AF &= bc\cos C/a, & BF &= c^2\cos B/a,\\ AF' &= c^2\cos A/b, & BF' &= ac\cos C/b. \end{aligned}\right\} \ldots\text{(viii)}.$$

If $\sigma_1 l_1$, $\sigma_2 l_2$ produced meet the side AC in h_1, h_2', and AB in h_1', h_2, then if K be put for $a^2+b^2+c^2$, we obtain

$$\sigma_1 h_1 = 2a^2 b\cos C/K, \quad \sigma_2 h_2 = 2a^2 c\cos B/K,$$

whence $\sigma_1 h_1 + \sigma_2 h_2 = 2a^3/K$, = (by I. x.)* twice the intercept made on BC by the T. R. circle.†

Also $\sigma_1 h_1' = 2abc\cos A/K = \sigma_2 h_2'$,
and therefore $\sigma_1 h_1' \sigma_2 h_2'$ is a parallelogram, $h_1'h_2'$ being bisected in the Lemoine point with like results for the other sides. ...(ix).

Again $$\frac{Ah_1'}{AB} = \frac{\sigma_1 h_1'}{BD}, \quad \frac{Ah_2'}{AC} = \frac{\sigma_2 h_2'}{CD},$$

hence $$Ah_1'.AB = Ah_2'.AC,$$

i.e. $h_1'h_2'$ is an anti-parallel to BC .... (x).

Since $Ah_1' = 2b^2c/K, \quad Ah_2' = 2bc^2/K,$

therefore

$$h_1'h_2' = 2abc/K (= 2DF' = 2ED' = 2FE' \text{ of I}) \ldots\text{(xi)}.$$

Similarly for the other sides ($j_1'j_2'$, $k_1'k_2'$ corresponding to $h_1'h_2'$), therefore

$$h_1'h_2' = j_1'j_2' = k_1'k_2',$$

i.e. these lines are diameters of the cosine-circle ......(xii); the equation of which is ($\mu = 1 + \cos A\cos B\cos C$),

$$\mu K(\alpha\beta\gamma + \ldots + \ldots)$$
$$= 2(a\alpha + \ldots + \ldots)[abc\sin B\sin C\cos A + \ldots + \ldots] \ldots\text{(xiii)}.$$

---

* I cite "The Triplicate-ratio circle," *Quar. Jour.*, vol. XIX., No. 76, as I.

† We readily obtain $\frac{h_1 h_1'}{b^2} = \frac{h_2 h_2'}{c^2}$, &c.

It is readily seen that $\sigma_1 j_1' j_2' h_1'$, $\sigma_2 j_2' j_1' h_2'$ together equal $j_1' j_2' k_2' k_1'$ .... (xiv).

If we draw AJ perpendicular to AC and equal Ah_1', then $\angle ACJ = \omega$; similar constructions for the other sides enable us to determine the Brocard-point in a second way, and in combination with (v) *supra* we have still other ways of finding them.

The equations to DM, and $D'N$ are respectively

$$\left.\begin{array}{l} a\alpha \cos A + b\beta \cos A - a\gamma \cos C = 0 \\ a\alpha \cos A - a\beta \cos B + c\gamma \cos A = 0 \end{array}\right\} \text{........(xv)},$$

and these lines intersect on

$$c\beta = b\gamma,$$

i.e. on the symmedian through A.

From (viii) we obtain

$$\Delta DEF = \Delta D'E'F' = 2\Delta ABC \cos A \cos B \cos C = \Delta LMN \text{(xvi)}.$$

The equation to the circle round DEF is

$$(a\beta\gamma \ldots) + (a\alpha + \ldots)\, T\,(ab \cos A \cos C\alpha$$
$$+ bc \cos B \cos A\beta + ca \cos C \cos B\gamma) = 0,$$

where

$$T = (a^3 c \cos A \cos C + b^3 a \cos B \cos A$$
$$+ c^3 b \cos C \cos B)/2a^2 b^2 c^2 \cos A \cos B \cos C$$
$$= R^2 [a^2 \sin 2A \sin 2C + b^2 \sin 2B \sin 2A + c^2 \sin 2C \sin 2B]/(\quad).$$

Similarly the equation to the circle $D'E'F'$ is

$$(a\beta\gamma + \ldots) + (a\alpha + \ldots)\, T'\,(ac \cos A \cos B\alpha$$
$$+ ba \cos B \cos C\beta + cb \cos C \cos A\gamma) = 0,$$

where

$$T' = R^2 [a^2 \sin 2A \sin 2B + b^2 \sin 2B \sin 2C$$
$$+ c^2 \sin 2C \sin 2A]/(\quad) \text{......(xvii)}.$$

From (xii), since the circle round $h_1' j_1' k_1'$, $h_2' j_2' k_2'$ is the cosine circle, these triangles have their sides perpendicular to the sides of ABC, and we see also that σ_1, σ_2 are the orthocentres of these "cosine" triangles. Hence, since K is the "cosine" centre, $K\sigma_1$, $K\sigma_2$ are perpendicular to OH and

$$\sigma_1\sigma_2 = 2\,OH \tan\omega = 2R\sqrt{(1 - 8 \sin A \cos B \cos C)} \tan\omega \text{...(xviii)}.$$

I collect together here a few results of interest.

The equation to $\sigma_1\sigma_2$ is, if $\nu^4 = a^4 + b^4 + c^4$,

$$a\alpha(a^2K - \nu^4) + b\beta(\quad) + c\gamma(\quad) = 0^* \text{......(xix)},$$

to $\sigma_1 G$ is

$$\left.\begin{array}{l} a\alpha(c^2 - a^2) + b\beta(a^2 - b^2) + c\gamma(b^2 - c^2) = 0, \\ \text{which passes through} \\ b^2/a, \quad c^2/b, \quad a^2/c, \quad (\mu_1), \\ (c^2 + a^2)/a, \quad (a^2 + b^2)/b, \quad (b^2 + c^2)/c \\ \text{and } 1/(c+a)a, \quad 1/(a+b)b, \quad 1/(b+c)c... \end{array}\right\} \text{....(xx)},$$

to $\sigma_2 G$ is

$$\left.\begin{array}{l} a\alpha(a^2 - b^2) + b\beta(b^2 - c^2) + c\gamma(c^2 - a^2) = 0, \\ \text{which passes through} \\ c^2/a, \quad a^2/b, \quad b^2/c, \quad (\mu_2), \\ (a^2 + b^2)/a, \quad (b^2 + c^2)/b, \quad (c^2 + a^2)/c, \\ \text{and } 1/(a+b)a, \quad 1/(b+c)b, \quad 1/(c+a)c, \end{array}\right\} \text{...(xxi)}.$$

The join of GK, $a\alpha(b^2 - c^2) + ... + ... = 0$, evidently passes through the middle point of the join of μ_1, μ_2.........(xxii), and the equation to $\mu_1\mu_2$ is

$$a\alpha(a^4 - b^2c^2) + ... + ... = 0 \text{ ....(xxiii)}.$$

Since $AFLE'$, &c., are parallelograms, therefore AL, FE', &c., mutually bisect each other.

Since $Cj_1' = a \cot C \tan\omega$, $Bj_1' = c \operatorname{cosec} B \tan\omega$, we get the perpendicular from Ω on $AL = \tan\omega\,(c \operatorname{cosec} B \sin\omega)$, *i.e.* Ω is Brocard-point of one cosine triangle, and Ω' of the other triangle .... (xxiv).

The lines $\sigma_1 j_1'$, $\sigma_2 j_2'$ intersect in

$$bc\cos A, \quad a^2\cos C, \quad a^2\cos B, \quad (W_1),$$

which is evidently on AL ....(xxv).

We note the following points in the figure, which will be of use in the sequel.

$$\left.\begin{array}{l} \left\{\begin{array}{llll} AG,\ B\sigma_1,\ CN & \text{meet in} & \pi_1 & (c\cos B,\ c\cos A,\ b\cos A) \\ BG,\ C\sigma_1,\ AL & \quad ,, & \pi_2 & (c\cos B,\ a\cos C,\ a\cos B) \\ CG,\ A\sigma_1,\ BM & \quad ,, & \pi_3 & (b\cos C,\ a\cos C,\ b\cos A) \end{array}\right. \\ \left\{\begin{array}{llll} AG,\ BM,\ C\sigma_2 & \quad ,, & \pi_1' & (b\cos C,\ c\cos A,\ b\cos A) \\ BG,\ CN,\ A\sigma_2 & \quad ,, & \pi_2' & (c\cos B,\ c\cos A,\ a\cos B) \\ CG,\ AL,\ B\sigma_2 & \quad ,, & \pi_3' & (b\cos C,\ a\cos C,\ a\cos B) \end{array}\right. \end{array}\right\}$$

.......................(xxvi),

* It is readily proved by rotating the figures that OH is perpendicular to $\sigma_1\sigma_2$.

$$\left.\begin{array}{l}\left\{\begin{array}{l}B\sigma_1,\ C\sigma_2 \text{ in } \tau_1(\tan A/a,\ \tan C/b,\ \tan B/c)\\ C\sigma_1,\ A\sigma_2 \text{ in } \tau_2(\tan C/a,\ \tan B/b,\ \tan A/c)\\ A\sigma_1,\ B\sigma_2 \text{ in } \tau_3(\tan B/a,\ \tan A/b,\ \tan C/c)\end{array}\right.\\ \left\{\begin{array}{l}C\sigma_1,\ B\sigma_2 \text{ in } \tau'_1(\cot A/\cos A,\ \cot C/\cos B,\ \cot B/\cos C)\\ A\sigma_1,\ C\sigma_2 \text{ in } \tau'_2(\cot C/\cos A,\ \cot B/\cos B,\ \cot A/\cos C)\\ B\sigma_1,\ A\sigma_2 \text{ in } \tau'_3(\cot B/\cos A,\ \cot A/\cos B,\ \cot C/\cos C)\end{array}\right.\end{array}\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxvii)},$$

$$\left.\begin{array}{l}\left\{\begin{array}{l}A\sigma_1,\ BG, \text{ in } v_1(\cot A/a,\ \cot C/b,\ \cot A/c)\\ B\sigma_1,\ CG, \text{ in } v_2(\cot B/a,\ \cot B/b,\ \cot A/c)\\ C\sigma_1,\ AG, \text{ in } v_3(\cot B/a,\ \cot C/b,\ \cot C/c)\end{array}\right.\\ \left\{\begin{array}{l}A\sigma_2,\ CG, \text{ in } v_1'(\cot A/a,\ \cot A/b,\ \cot B/c)\\ B\sigma_2,\ AG, \text{ in } v_2'(\cot C/a,\ \cot B/b,\ \cot B/c)\\ C\sigma_2,\ BG, \text{ in } v_3'(\cot C/a,\ \cot A/b,\ \cot C/c)\end{array}\right.\end{array}\right\}\ldots\text{(xxviii)}.$$

If $A_1B_1C_1$ is the first Brocard triangle, then AA_1, BB_1, CC_1 bisect EF', FD', DE' respectively (and intersect as is well-known on Kiepert's hyperbola) .... (xxix).

The pole of $\sigma_1\sigma_2$ with regard to the circum-circle is

$$\left.\begin{array}{l} a^2[bc - a^2\cos(B-C)],\ \ldots,\ \ldots, \\ \text{and therefore lies on the well-known line} \\ bc\alpha + ca\beta + ab\gamma = 0 \end{array}\right\}\ldots\text{(xxx)},$$

and of $\varepsilon_1\varepsilon_2$, lies on

$$\alpha(b^2+c^2)/a + \ldots + \ldots = 0,$$

i.e. on the radical axis of the circum- and "T.R" circles .... (xxxi).

Assume $\sigma_1BC = \phi_1$, $\sigma_2CB = \phi_1'$,

then $$\frac{\sin(B-\phi_1)}{\sin\phi_1} = \frac{b\cos A}{a\cos B},$$

whence $$\cot\phi_1 = \tan B + \cot B - \cot C,$$

and $$\cot\phi_1' = \tan C + \cot C - \cot B;$$

$$\left.\begin{array}{l}\text{therefore}\quad \cot\phi_1 + \cot\phi_1' = \tan B + \tan C,\\ \text{and}\quad \cot\phi_1 + \cot\phi_2 + \cot\phi_3 = \tan A + \tan B + \tan C\\ \quad = \tan A\tan B\tan C = \cot\phi'_1 + \cot\phi'_2 + \cot\phi'_3\end{array}\right\}\ldots\text{(xxxii)}.$$

If $$\sigma_2 BC = \psi_1,\ \sigma_1 CB = \psi'_1,$$

then $$\frac{\sin(B-\psi_1)}{\sin\psi_1} = \frac{a\cos B}{b\cos C},$$

$$\left.\begin{array}{lr} \text{whence} & \cot\psi_1 = 2\cot B + \cot^2 B\tan C, \\ \text{and} & \cot\psi_1' = 2\cot C + \cot^2 C\tan B, \\ \text{therefore} & \Sigma\cot\psi.\cot\psi' = 4\cot^2\omega - 3. \end{array}\right\} \ldots\ldots \text{(xxxiii)}.$$

From (viii) we see that $AE.AM = b^2\cos^2 A = AN^2$, therefore circle round NEM touches AB at N; similar results hold good for the other sides ....(xxxiv).

The equations to the circles $LE'M$, LFN are respectively

$$x^2 + y^2 = cy\cos C/\sin A,$$

and $$x^2 + y^2 = by\cos B/\sin A \ldots\ldots\ldots\ldots \text{(xxxv)},$$

where CL, LA are the axes; whence, if D, D' are their diameters, we have $D + D' = R\cos(B - C)$.

The trilinear equations to the circles in (xxxiv) are

$$\left.\begin{array}{ll} FLN, & aC = L[b\cos C\cos A\alpha + c\cos^2 B\beta + b\cos^2 C\gamma] \\ DML, & bC = L[c\cos^2 A\alpha + c\cos A\cos B\beta + a\cos^2 C\gamma] \\ ENM, & cC = L[b\cos^2 A\alpha + a\cos^2 B\beta + a\cos B\cos C\gamma] \\ \text{and } E'ML, & aC = L[c\cos B\cos A\alpha + c\cos^2 B\beta + b\cos^2 C\gamma] \end{array}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxxvi)},$$

$$F'NM,\quad bC = L[c\cos^2 A\alpha + a\cos C\cos B\beta + a\cos^2 C\gamma],$$

$$D'LN,\quad cC = L[b\cos^2 A\alpha + a\cos^2 B\beta + b\cos A\cos C\gamma],$$

if $$C \equiv a\beta\gamma + b\gamma\alpha + c\alpha\beta \text{ and } L \equiv a\alpha + b\beta + c\gamma.$$

From (xxxv) we see that if ρ_1, ρ_2, ρ_3; $\rho'_1, \rho'_2, \rho'_3$ are the two sets of radii, then

$$\left.\begin{array}{c} \rho_1\rho_2\rho_3 = abc\cot A\cot B\cot C/8 = \rho'_1\rho'_2\rho'_3, \\ a\rho_1 + b\rho_2 + c\rho_3 = 2R^2\sin A\sin B\sin C = a\rho'_1 + b\rho'_2 + c\rho'_3. \end{array}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxxvii)}.$$

Some general properties of the cubics given by the equation

$$C_3 \equiv (a\alpha + b\beta + c\gamma)\left(\frac{1}{a\alpha} + \frac{1}{b\beta} + \frac{1}{c\gamma}\right) = k^2$$

are given in the *Messenger of Mathematics* (vol. II., 1864,

p. 116) and in the *Reprint from the Educational Times*, (vol. v., p. 38); we propose here to consider the cubic

$$C_3 = \cot\omega \tan A \tan B \tan C = k' \; \ldots \; (\text{xxxviii}),$$

which passes through the points of reference, through σ_1, σ_2, and through the orthocentre.

The equation can be put into the form

$$(a\alpha + b\beta + c\gamma)(bc\beta\gamma + ca\gamma\alpha + ab\alpha\beta) = k'abc\alpha\beta\gamma \ldots (\text{xxxix}),$$

from whence it is seen that it touches the minimum circum-ellipse $\beta\gamma/a + \gamma\alpha/c + \alpha\beta/c = 0$ at the points of reference.

The centroid of the triangle is the centre of this ellipse, and HA, HB, HC, which are normals to the ellipse, are also normals to the cubic, and, from above statement, are drawn from a point on the cubic.

The six points H, τ_1, τ_2, τ_3 (xxvii), and

$$\begin{array}{llll} \tau, & \tan B/a, & \tan C/b, & \tan A/c \; \ldots\ldots (\text{xl}),^* \\ \tau', & \tan C/a, & \tan A/b, & \tan B/c, \end{array}$$

are manifestly on C_3, and they also lie on the ellipse

$$ab\alpha\beta + bc\beta\gamma + ca\gamma\alpha = \cot\omega \cot A \cot B \cot C$$
$$= (a\alpha + b\beta + c\gamma)^2 \ldots\ldots\ldots\ldots (\text{xli}),$$

which is concentric and similarly placed with the above minimum ellipse.

The cubic also passes through the six points

$$\left.\begin{array}{llll} \beta_1, & \cot A/a, & \cot B/b, & \cot C/c, \\ \beta_2, & \cot A/a, & \cot C/b, & \cot B/c, \\ \beta_3, & \cot B/a, & \cot A/b, & \cot C/c, \\ \beta_4, & \cot B/a, & \cot C/b, & \cot A/c, \\ \beta_5, & \cot C/a, & \cot A/b, & \cot B/c, \\ \beta_6, & \cot C/a, & \cot B/b, & \cot A/c. \end{array}\right\} \ldots\ldots (\text{xlii}),$$

which lie also on the ellipse

$$ab\alpha\beta + bc\beta\gamma + ca\gamma\alpha = \tan^2\omega\,(a\alpha + b\beta + c\gamma)^2$$

concentric and similarly placed with the minimum ellipse.

March, 1887.

---

* These points are easily constructed in a different way from that indicated in (xxvii): for $\sigma_1\,(\alpha_1)$, $\sigma_2\,(\alpha_2)$, $\tau\,(\alpha)$, $\tau'\,(\alpha')$ we have the relation between coordinates to be $a^2\alpha\alpha_1 = a^2\,(\alpha'\alpha_2) = b^2\beta\beta_1 = \&c.$

NOTE ON THE MULTIPLICATION OF NONIONS.

By *G. G. Morrice*.

THE object of the present note is to present the multiplication table of nonions in its proper form. We have m and n for our two fundamental ternary matrices with the condition $nm = \rho mn$, where ρ is a primitive cube root of unity, $m^3 = n^3 = 1$.

1	m	m^2	n	mn	m^2n	n^2	mn^2	m^2n^2
m	m^2	1	ρmn	ρm^2n	ρn	ρ^2mn^2	$\rho^2m^2n^2$	ρ^2n^2
m^2	1	m	ρ^2m^2n	ρ^2n	ρ^2mn	ρm^2n^2	ρn^2	ρmn^2
n	mn	m^2n	n^2	mn^2	m^2n^2	1	m	m^2
mn	m^2n	n	ρmn^2	ρm^2n^2	ρn^2	ρ^2m	ρ^2m^2	ρ^2
m^2n	n	mn	$\rho^2m^2n^2$	ρ^2n^2	ρ^2mn^2	ρm^2	1	ρm
n^2	mn^2	m^2n^2	1	m	m^2	n	mn	m^2n
mn^2	m^2n^2	n^2	ρm	ρm^2	ρ	ρ^2mn	ρ^2m^2n	ρn^2
m^2n^2	n^2	mn^2	ρ^2m^2	ρ^2	ρ^2m	ρm^2n	ρn	ρmn

Now in any such multiplication for a group of operators, and in particular for substitutions (*vide* Dyck's *Gruppentheoretische Studien II.*), it is important to consider the permutations of the elements of the initial row which leads to any one of the following rows. In the present case we have clearly a non-primitive group, the substitutions which interchange the triads $(1, m, m^2)$, (n, mn, m^2n), (n^2, mn^2, m^2n^2) being cyclic, and also the substitutions which interchange the letters within the cycles. We may call the substitutions

$$\begin{array}{ccc} 1, & s, & s^2, \\ S, & Ss, & Ss^2, \\ S^2, & S^2s, & S^2s^2. \end{array}$$

Let us form the matrix M,

$$\begin{array}{ccc} 1, & m, & m^2, \\ n, & mn, & m^2n, \\ n^2, & mn^2, & m^2n^2; \end{array}$$

s will effect a cyclic interchange of columns, S of rows.

Moreover, the simplest form of m is

$$m' = \begin{pmatrix} 1, & 0, & 0 \\ 0, & \rho, & 0 \\ 0, & 0, & \rho^2 \end{pmatrix}, \qquad m'^2 = \begin{pmatrix} 1, & 0, & 0 \\ 0, & \rho^2, & 0 \\ 0, & 0, & \rho \end{pmatrix},$$

and it is easy to verify that, for example, the elements of the second row are those of the matrix formed by multiplying m' by $s(M)$, the third row m'^2 by $s^2(M)$, and so on.

June 21, 1887.

VORTICES IN A COMPRESSIBLE FLUID.

By *Charles Chree, M.A.*, Fellow of King's College, Cambridge.

THE following paper contains certain applications of the equations of vortex motion in two dimensions to a compressible fluid. The equations of vortex motion in an infinite fluid are*

$$\left.\begin{aligned} u &= \frac{dP}{dx} + \frac{dN}{dy} - \frac{dM}{dz} \\ v &= \frac{dP}{dy} + \frac{dL}{dz} - \frac{dN}{dx} \\ w &= \frac{dP}{dz} + \frac{dM}{dx} - \frac{dL}{dy} \end{aligned}\right\} \qquad (1).$$

If $$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \equiv \theta \qquad (2),$$

then $$P = -\frac{1}{4\pi}\iiint \frac{\theta'}{r}\, dx'dy'dz' \qquad (3)$$

gives the value of the function P at a point (x, y, z) at a distance r from the point (x', y', z'), where θ has the value θ', the integration extending through all space. If $\frac{d}{dt}$ denote partial differentiation, and $\frac{\partial}{\partial t}$ differentiation following the fluid, the equation of continuity is

$$\frac{d\rho}{dt} + \frac{d\rho u}{dx} + \frac{d\rho v}{dy} + \frac{d\rho w}{dz} = 0,$$

or $$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \theta = 0.$$

---

* See Lamb's *Motion of Fluids*, pp. 150, 151, for the meaning of L, M, N, &c.

Thus $$P=\frac{1}{4\pi}\iiint\frac{1}{r}\,\frac{1}{\rho}\,\frac{\partial\rho}{\partial t}\,dx'dy'dz' \qquad\text{..............(4).}$$

From the form of this integral and of the equations of motion it follows that the direction and numerical measure of the velocity, at any point dependent on the function P, are the same as the direction and numerical measure of the resultant force at that point in an imaginary gravitating system, whose density at (x', y', z') equals $\frac{1}{4\pi\rho}\frac{\partial\rho}{\partial t}$. Thus the velocity due to variation of density in the fluid can be deduced from the theory of attractions.

Suppose now all the vortices to be straight and parallel to the axis of z, then the velocity, so far as depends on the vortices, is independent of z, *i.e.* is in two dimensions. The functions L and M vanish, and

$$N^{*}=-\frac{1}{\pi}\iint\zeta'\log r\,dx'dy';$$

or, if σ' denote the cross-section of the elementary vortex filament whose vorticity is ζ', and Σ denote summation of such elements,

$$N=-\frac{1}{\pi}\,\Sigma\log r\,\zeta'\sigma' \qquad\text{..................(5).}$$

The motion depending on P will also be independent of z, provided $\frac{1}{\rho}\frac{\partial\rho}{\partial t}$ be independent of z. This is the case if $w=0$ and $\rho=\rho_0 f(z)$, where ρ_0 is the density at the plane $z=0$, which may be any function of x and y, while $f(z)$ denotes any function of z which is unity when $z=0$. If ρ' be the density over the elementary cross-section σ', the equation of continuity becomes

$$\sigma'\rho'=\text{constant.}$$

Thus $$\frac{1}{\rho'}\frac{\partial\rho'}{\partial t}=-\frac{1}{\sigma'}\frac{\partial\sigma'}{\partial t},$$

and we simply get, in place of (4),

$$P=\frac{1}{2\pi}\,\Sigma\log r\,\frac{\partial\sigma'}{\partial t} \qquad\text{.....................(6).}$$

---

* Lamb, p. 168, Equation (33),

In what follows σ always refers to a definite column of fluid, thus no difference of meaning exists between $\frac{\partial\sigma}{\partial t}$ and $\frac{d\sigma}{dt}$, and in our further treatment it will be more convenient to employ the latter term. The same remark applies to the variation of the radius or of the coordinates of the centre of a particular vortex column.

The kind of motion just considered is consistent with the existence of any external force parallel to oz, and any function whatsoever of z. For, denoting this force by Z, and supposing the velocity everywhere parallel to xy, we have

$$Z - \frac{1}{\rho}\frac{dp}{dz} = 0.*$$

If then $p = k\rho^\gamma$, where k and γ are any constants, we get a solution of the form $\rho = \rho_0 f(z)$, where ρ_0 may vary from point to point of the plane $z = 0$, and $f(z) = 1$ when $z = 0$.

In the expressions for N and P, σ' may be supposed to refer to elementary areas in the plane of xy. The fluid in which the density is varying need not be the same portion as possesses vorticity, but in the following applications we shall suppose this to be the case. In the case of a vortex of any form of cross-section, the velocity, at a distance from the vortex which is great compared to the greatest diameter of the section, is got with sufficient accuracy by taking the entire section for σ', and supposing ζ' to be the mean value of the vorticity and r the distance from the centroid of the cross-section. If ρ' be the mean density the equation of continuity is strictly $\sigma'\rho' = $ constant, and thus the expression (6) for P is at least as accurate as the expression (5) for N. The more nearly the cross-section is circular, and the vorticity and density constant throughout, or functions only of the distance from the centre of the circle, the more legitimate is this treatment.

No change in the method is required, though the fluid be limited to the positive side of xy by a rigid boundary coinciding with that plane. It is only necessary to treat the vortices as extending to infinity, with a composition the same at $-z$ as at $+z$, in an imaginary fluid on the negative side of xy to ensure that the velocity is everywhere parallel to xy. In like manner the case of a vortex in presence of a plane parallel to oz, or in an angle formed by two such planes, can

* Lamb, p. 5, Equation (2).

be treated exactly as in an incompressible fluid by the aid of images. Each image is to have the same cross-section at every instant as the real vortex, and the consequent velocity due to variation of density will in every case be parallel to the boundary planes.

Consider now a single straight vortex, parallel to oz, and let m be its strength and σ its cross-section at time t; then the components of the velocity at a point in the surrounding fluid, whose coordinates are x and y relative to axes through the centre of the vortex fixed in direction, are given by

$$\left.\begin{aligned} u &= -\frac{my}{\pi r^2} + \frac{1}{2\pi}\frac{d\sigma}{dt}\frac{x}{r^2} \\ v &= \frac{mx}{\pi r^2} + \frac{1}{2\pi}\frac{d\sigma}{dt}\frac{y}{r^2} \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

For our special object we require to determine the motion of two such vortices, whose mutual distance is supposed great compared to the diameters of their cross-sections. Let their strengths be m_1, m_2, and at time t let their cross-sections be σ_1, σ_2, and the coordinates of their centres (x_1, y_1) and (x_2, y_2); then, if $r^2 \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2$, we have

$$\left.\begin{aligned} \frac{dx_2}{dt} &= -\frac{m_1(y_2-y_1)}{\pi r^2} + \frac{1}{2\pi}\frac{d\sigma_1}{dt}\frac{x_2-x_1}{r^2} \\ \frac{dy_2}{dt} &= \frac{m_1(x_2-x_1)}{\pi r^2} + \frac{1}{2\pi}\frac{d\sigma_1}{dt}\frac{y_2-y_1}{r^2} \\ \frac{dx_1}{dt} &= -\frac{m_2(y_1-y_2)}{\pi r^2} + \frac{1}{2\pi}\frac{d\sigma_2}{dt}\frac{x_1-x_2}{r^2} \\ \frac{dy_1}{dt} &= \frac{m_2(x_1-x_2)}{\pi r^2} + \frac{1}{2\pi}\frac{d\sigma_2}{dt}\frac{y_1-y_2}{r^2} \end{aligned}\right\} \quad\ldots\ldots\ldots(8).$$

From these equations we get

$$(x_2 - x_1)\frac{d}{dt}(x_2 - x_1) + (y_2 - y_1)\frac{d}{dt}(y_2 - y_1) = \frac{1}{2\pi}\frac{d}{dt}(\sigma_1 + \sigma_2).$$

Thus if when $t = 0$, $r = c$, $\sigma_1 = {}_0\sigma_1$, and $\sigma_2 = {}_0\sigma_2$, we get

$$r^2 = c^2 + \frac{1}{\pi}(\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2) \quad\ldots\ldots\ldots\ldots(9).$$

These equations also give

$$\frac{d}{dt}\left(\frac{y_2 - y_1}{x_2 - x_1}\right) = \frac{m_1 + m_2}{\pi(x_2 - x_1)^2}.$$

But if ε denote the inclination to ox of the line joining the centres of the vortices, $y_2 - y_1 = r\sin\varepsilon$ and $x_2 - x_1 = r\cos\varepsilon$, and the above equation becomes

$$\frac{d\varepsilon}{dt} = \frac{m_1 + m_2}{\pi r^2}.$$

Thus, if $\varepsilon = 0$ when $t = 0$, we find

$$\varepsilon = \frac{m_1 + m_2}{\pi}\int_0^t \frac{dt}{c^2 + \frac{1}{\pi}(\sigma_1 + \sigma_2 - {}_0\sigma_1 - {}_0\sigma_2)} \quad \text{.....(10).}$$

The case of a vortex before an infinite plane is included in this solution. We have only to take $\frac{1}{2}c$ as the distance of the vortex from the plane at the time $t = 0$, and make $m_2 = -m_1$, $\sigma_2 = \sigma_1$, and ${}_0\sigma_2 = {}_0\sigma_1$. In this case ε remains zero.

The stability of the circular form in one and in two straight vortices has been considered, under the title of *Linked Vortices*, by Prof. J. J. Thomson, in his "Motion of Vortex Rings."* The main object of this paper is to extend his treatment to a compressible fluid. To render more easy comparison with Prof. Thomson's results, I have followed his notation and method so far as possible.

Consider first a single, approximately circular, vortex whose section is the same for all values of z, and let the radius of the cross section making an angle θ with a fixed direction be given at time t by

$$R = a + \alpha_n \cos n\theta + \beta_n \sin n\theta \text{.............(11),}$$

the last two terms being the types of any number of pairs of terms, while α_n, β_n are supposed small compared with a.

Suppose the vorticity and the density to be at any instant the same at all points in the cross section. At external points we have approximately, as for a vortex of truly circular cross section, the functions

$$N = -\frac{m}{\pi}\log r, \qquad P = \frac{1}{2\pi}\frac{d\sigma}{dt}\log r.$$

Assume for the fluid outside the vortex

$$\left.\begin{aligned} N &= C - \frac{m}{\pi}\log r + (A_n\cos n\theta + B_n \sin n\theta)\left(\frac{a}{r}\right)^n \\ P &= D + \frac{1}{2\pi}\frac{d\sigma}{dt}\log r + (E_n \cos n\theta + F_n \sin n\theta)\left(\frac{a}{r}\right)^n \end{aligned}\right\}\text{...(12),}$$

* Part III., p. 71.

and for the fluid inside the vortex

$$\left.\begin{aligned} N' &= C' - \tfrac{1}{2}\frac{mr^2}{\pi a^2} + (A'_n \cos n\theta + B'_n \sin n\theta)\left(\frac{r}{a}\right)^n \\ P' &= D' + \frac{1}{4\pi a^2}\frac{d\sigma}{dt} r^2 + (E'_n \cos n\theta + F'_n \sin n\theta)\left(\frac{r}{a}\right)^n \end{aligned}\right\} \ldots(13),$$

where C, D, C', D' are absolute constants, while A_n, B_n, &c. denote, as do α_n, β_n, functions of the time. The values of N and N', and of P and P', must agree at the surface of the vortex, *i.e.*, when $r = a + \alpha_n \cos n\theta + \beta_n \sin n\theta$. We thus get, neglecting products of small quantities,

$$A'_n = A_n, \quad B'_n = B_n, \quad E'_n = E_n, \quad F'_n = F_n.$$

We must also have the velocities, radial and tangential, the same for points just inside and just outside the vortex; therefore, when $r = a + \alpha_n \cos n\theta + \beta_n \sin n\theta$,

$$\frac{1}{r}\frac{dN}{d\theta} + \frac{dP}{dr} = \frac{1}{r}\frac{dN'}{d\theta} + \frac{dP'}{dr}$$

and
$$-\frac{dN}{dr} + \frac{1}{r}\frac{dP}{d\theta} = -\frac{dN'}{dr} + \frac{1}{r}\frac{dP'}{d\theta}.$$

Neglecting terms such as $A_n \alpha_n$, it is however obvious, if $A'_n = A_n$, &c., that at the surface considered

$$\frac{1}{r}\frac{dN}{d\theta} = \frac{1}{r}\frac{dN'}{d\theta} \quad \text{and} \quad \frac{1}{r}\frac{dP}{d\theta} = \frac{1}{r}\frac{dP'}{d\theta};$$

thus we are left with

$$\frac{dN}{dr} = \frac{dN'}{dr}, \qquad \frac{dP}{dr} = \frac{dP'}{dr}.$$

Equating coefficients of $\cos n\theta$ and of $\sin n\theta$, we get from these equations

$$A_n = \frac{m}{\pi a n}\alpha_n, \qquad B_n = \frac{m}{\pi a n}\beta_n,$$

$$E_n = -\frac{1}{2\pi a n}\frac{d\sigma}{dt}\alpha_n, \quad F_n = -\frac{1}{2\pi a n}\frac{d\sigma}{dt}\beta_n.$$

Thus outside the vortex

$$\left.\begin{aligned} N &= C - \frac{m}{\pi}\log r + \frac{m}{\pi a n}(\alpha_n \cos n\theta + \beta_n \sin n\theta)\left(\frac{a}{r}\right)^n \\ P &= D + \frac{1}{2\pi}\frac{d\sigma}{dt}\log r - \frac{1}{2\pi a n}\frac{d\sigma}{dt}(\alpha_n \cos n\theta + \beta_n \sin n\theta)\left(\frac{a}{r}\right)^n \end{aligned}\right\}$$

..................(14).

But from (11)

$$\frac{dR}{dt} = \frac{da}{dt} + \frac{d\alpha_n}{dt}\cos n\theta + \frac{d\beta_n}{dt}\sin n\theta$$
$$- n\Theta\,(\alpha_n \sin n\theta - \beta_n \cos n\theta) \ldots\ldots (15),$$

where Θ is the angular velocity round the axis of the vortex of the fluid at its surface. Thus

$$\Theta = -\frac{1}{r}\frac{dN}{dr} + \frac{1}{r^2}\frac{dP}{d\theta} = \frac{m}{\pi a^2} + \text{terms in } \alpha_n \text{ and } \beta_n.$$

Thus in (15) we may take $\Theta = \dfrac{m}{\pi a^2}$.

We have also a second value for the radial velocity given by

$$\frac{dR}{dt} = \frac{1}{r}\frac{dN}{d\theta} + \frac{dP}{dr}.$$

Noticing that $\dfrac{1}{2\pi a}\dfrac{d\sigma}{dt} = \dfrac{da}{dt}$, this last value of $\dfrac{dR}{dt}$ can be reduced to

$$-\frac{m}{\pi a^2}(\alpha_n \sin n\theta - \beta_n \cos n\theta) + \frac{da}{dt},$$

which must be identical with (15). Equating the coefficients of $\cos n\theta$ and of $\sin n\theta$, we find

$$\left.\begin{aligned} \frac{d\alpha_n}{dt} &= -\beta_n (n-1)\frac{m}{\pi a^2} \\ \frac{d\beta_n}{dt} &= \quad \alpha_n (n-1)\frac{m}{\pi a^2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (16),$$

i.e. $\qquad \sigma\dfrac{d\alpha_n}{at} = -m(n-1)\beta_n, \quad \sigma\dfrac{d\beta_n}{dt} = m(n-1)\alpha_n.$

Thus $\qquad \alpha_n\dfrac{d\alpha_n}{dt} + \beta_n\dfrac{d\beta_n}{dt} = 0,$

whence $\qquad \alpha_n^2 + \beta_n^2 = \text{constant} \ldots\ldots\ldots\ldots\ldots\ldots (17).$

Also, eliminating β_n from (16), we get

$$\sigma^2\frac{d^2\alpha_n}{dt^2} + \sigma\frac{d\sigma}{dt}\frac{d\alpha_n}{dt} + m^2(n-1)^2\alpha_n = 0.$$

Multiplying up by $2\frac{d\alpha_n}{dt}$, and integrating, we find

$$\sigma^2\left(\frac{d\alpha_n}{dt}\right)^2+m^2(n-1)^2\alpha_n^2=\text{constant}\ldots\ldots\ldots(18).$$

If then α_n and $\frac{d\alpha_n}{dt}$ vanish initially they always do so, the same is obviously true of β_n. Suppose initially $\frac{d\alpha_n}{dt}=0$, and so from (16) $\beta_n=0$, while $\alpha_n={}_0\alpha_n$, and so $\frac{d\beta_n}{dt}={}_0\alpha_n\frac{m(n-1)}{\pi a_0^2}$, where a_0 is the initial value of a; then from (18)

$$\left.\begin{aligned}\alpha_n&={}_0\alpha_n\cos\left\{m(n-1)\int_\sigma^t\frac{dt}{\sigma}\right\};\\ \text{similarly}\qquad \beta_n&={}_0\alpha_n\sin\left\{m(n-1)\int_0^t\frac{dt}{\sigma}\right\}\end{aligned}\right\}\ldots\ldots\ldots\ldots(19).$$

Thus the form of the cross-section at time t is given by

$$R=a+{}_0\alpha_n\cos\left\{n\theta-m(n-1)\int_0^t\frac{dt}{\sigma}\right\}\ldots\ldots\ldots(20).$$

The section thus remains approximately circular, and the disturbance in shape travels round the cylinder in the gradually varying times given by

$$m(n-1)\int_0^t\frac{dt}{\sigma}=2i\pi,$$

where i is an integer.

If we suppose

$$\frac{d\sigma}{dt}=\text{constant}={}_0\sigma\gamma,$$

we get

$$R=a+{}_0\alpha_n\cos\left\{n\theta-\frac{m(n-1)}{{}_0\sigma\gamma}\log(1+\gamma t)\right\}.$$

The period of the ith revolution of the disturbance is given by

$$T_i=\frac{1}{\gamma}\left\{e^{\frac{2\pi_0\sigma\gamma}{m(n-1)}}-1\right\}e^{\frac{2(i-1)\pi_0\sigma\gamma}{m(n-1)}},$$

and so increases with i if γ be positive, *i.e.* if the cross-section of the vortex be increasing.

If $\sigma=\text{constant}=\pi a^2$, and $m=\omega\pi a^2$ and the time be properly chosen, the result (20) is identical with that of Prof. Thomson on his page 74.

We shall next consider the case of two vortices of the same nature as the last. Let their strengths be m_1, m_2, and their cross sections at time t, σ_1 and σ_2, the distance between their axes being then c. Suppose the radii of their cross sections given by

$$\left.\begin{aligned} R &= a + \alpha_n \cos n\theta + \beta_n \sin n\theta \\ R' &= b + \alpha'_n \cos n\theta' + \beta'_n \sin n\theta' \end{aligned}\right\} \qquad (21).$$

Let N_1, P_1 be the functions for the first at an external point, and N_2, P_2 for the second. Then, if r denote the distance of a point from the axis of the first vortex, r' that from the axis of the second, we get, as in (14),

$$\left.\begin{aligned} N_1 &= C_1 - \frac{m_1}{\pi}\log r + \frac{m_1}{\pi a n}\{\alpha_n \cos n\theta + \beta_n \sin n\theta\}\left(\frac{a}{r}\right)^n \\ P_1 &= D_1 + \frac{1}{2\pi}\frac{d\sigma_1}{dt}\log r \\ &\qquad - \frac{1}{2\pi a n}\frac{d\sigma_1}{dt}\{\alpha_n \cos n\theta + \beta_n \sin n\theta\}\left(\frac{a}{r}\right)^n \\ N_2 &= C_2 - \frac{m_2}{\pi}\log r' + \frac{m_2}{\pi b n}\{\alpha'_n \cos n\theta' + \beta'_n \sin n\theta'\}\left(\frac{b}{r}\right)^n \\ P_2 &= D_2 + \frac{1}{2\pi}\frac{d\sigma_2}{dt}\log r' \\ &\qquad - \frac{1}{2\pi b n}\frac{d\sigma_2}{dt}\{\alpha'_n \cos n\theta' + \beta'_n \sin n\theta'\}\left(\frac{b}{r}\right)^n \end{aligned}\right| \qquad (22).$$

Let ε denote the angle the line joining the axes of the two vortices makes with the initial line, then, following Prof. Thomson's method, we find the following expressions for N_1 and P_1 in terms of θ' and ε:

$$\begin{aligned} N_1 = C_1 &- \frac{m_1}{\pi}\left\{\log c + \frac{r'}{c}\cos(\theta'-\varepsilon) - \tfrac{1}{2}\frac{r'^2}{c^2}\cos 2(\theta'-\varepsilon) + \dots\right\} \\ &+ \frac{m_1}{\pi n}\frac{a^{n-1}}{c^n}\left[(\alpha_n \cos n\varepsilon + \beta_n \sin n\varepsilon)\left\{1 - \frac{nr'}{c}\cos(\theta'-\varepsilon) + \dots\right\}\right. \\ &\left. + n(\beta_n \cos n\varepsilon - \alpha_n \sin n\varepsilon)\left\{\frac{r'}{c}\sin(\theta'-\varepsilon) - \dots\right\}\right] \qquad (23), \end{aligned}$$

$$\begin{aligned} P_1 = D_1 &+ \frac{1}{2\pi}\frac{d\sigma_1}{dt}\left\{\log c + \frac{r'}{c}\cos(\theta'-\varepsilon) - \tfrac{1}{2}\frac{r'^2}{c^2}\cos 2(\theta'-\varepsilon) + \dots\right\} \\ &- \frac{1}{2\pi n}\frac{d\sigma_1}{dt}\frac{a^{n-1}}{c^n}\left[(\alpha_n \cos n\varepsilon + \beta_n \sin n\varepsilon)\left\{1 - \frac{nr'}{c}\cos(\theta'-\varepsilon) + \dots\right\}\right. \\ &\left. + n(\beta_n \cos n\varepsilon - \alpha_n \sin n\varepsilon)\left\{\frac{r'}{c}\sin(\theta'-\varepsilon) - \dots\right\}\right] \qquad (24). \end{aligned}$$

Since we suppose a/c small we require to retain to our present degree of approximation only α_1, α_2, β_1, and β_2.

Let $\mathfrak{R}$ denote the radial velocity of a point on the second vortex, and $b\Theta$ the velocity perpendicular to the radius vector, both being taken relative to the centre of that vortex. Then, from (21),

$$\mathfrak{R} = \frac{db}{dt} + \frac{d\alpha'_n}{dt}\cos n\theta' + \frac{d\beta'_n}{dt}\sin n\theta'$$

$$- n\Theta\,(\alpha'_n \sin n\theta' - \beta'_n \cos n\theta') \ldots\ldots\ldots (25),$$

where to the present degree of approximation

$$\Theta = \frac{m_2}{\pi b^2}.$$

But we have also, putting $r' = b$ after differentiation,

$$\mathfrak{R} = \frac{1}{r'}\frac{dN_2}{d\theta'} + \frac{dP_2}{dr'} + \frac{1}{r'}\frac{dN_1}{d\theta'} - \frac{m_1}{\pi c}\sin(\theta' - \epsilon)$$

$$+ \frac{dP_1}{dr'} - \frac{1}{2\pi c}\frac{d\sigma_1}{dt}\cos(\theta' - \epsilon).$$

The terms in m_1 and in $\dfrac{d\sigma_1}{dt}$ are introduced as it is the velocity relative to the centre of the second vortex that is being considered. We thus get, the terms in n being of course merely typical,

$$\mathfrak{R} = -\frac{m_2}{\pi b^2}(\alpha'_n \sin n\theta' - \beta'_n \cos n\theta') + \frac{1}{2\pi b}\frac{d\sigma_2}{dt}$$

$$- \frac{m_1}{\pi c^2}\, b \sin 2(\theta' - \epsilon) \ldots$$

$$+ \frac{m_1}{\pi}\frac{a^{n-1}}{c^{n+1}}\{(\alpha_n \cos n\epsilon + \beta_n \sin n\epsilon)\sin(\theta' - \epsilon)$$

$$+ (\beta_n \cos n\epsilon - \alpha_n \sin n\epsilon)\cos(\theta' - \epsilon) \ldots\} - \frac{1}{2\pi c^2}\frac{d\sigma_1}{dt}\, b\cos 2(\theta' - \epsilon) \ldots$$

$$+ \frac{1}{2\pi}\frac{d\sigma_1}{dt}\frac{a^{n-1}}{c^{n+1}}\{(\alpha_n \cos n\epsilon + \beta_n \sin n\epsilon)\cos(\theta' - \epsilon)$$

$$- (\beta_n \cos n\epsilon - \alpha_n \sin n\epsilon)\sin(\theta' - \epsilon) \ldots\} \ldots\ldots (26).$$

These two values for $\mathfrak{K}$ must be identical. The terms independent of θ' are so since $\frac{d\sigma_2}{dt}=2\pi b\frac{db}{dt}$, and the identity extends to the coefficients of every sine and cosine of multiples of θ'.

Equating coefficients of $\cos\theta'$, we get, after reduction,

$$\frac{d\alpha'_1}{dt}=\frac{m_1}{\pi c^2}(\beta_1\cos 2\varepsilon-\alpha_1\sin 2\varepsilon)+\frac{1}{2\pi c^2}\frac{d\sigma_1}{dt}(\alpha_1\cos 2\varepsilon+\beta_1\sin 2\varepsilon);$$

but to our present degree of approximation terms of orders $\frac{\alpha m}{c^2}$ or $\frac{\alpha}{c^2}\frac{d\sigma}{dt}$ are negligible, so $\frac{d\alpha'_1}{dt}=0$; similarly we get $\frac{d\beta'_1}{dt}=0$. Thus α'_1, β'_1 if originally zero remain so, whether the fluid be compressible or not.

To the same degree of approximation we find, from equating the coefficients of $\cos 2\theta'$ and of $\sin 2\theta'$ in (25) and (26),

$$\frac{d\alpha'_2}{dt}+\frac{m_2}{\pi b^2}\beta'_2-\frac{m_1 b}{\pi c^2}\sin 2\varepsilon+\frac{1}{2\pi}\frac{b}{c^2}\frac{d\sigma_1}{dt}\cos 2\varepsilon=0\ldots\ldots..(27),$$

$$\frac{d\beta'_2}{dt}-\frac{m_2}{\pi b^2}\alpha'_2+\frac{m_1 b}{\pi c^2}\cos 2\varepsilon+\frac{1}{2\pi}\frac{b}{c^2}\frac{d\sigma_1}{dt}\sin 2\varepsilon=0\ldots\ldots..(28).$$

It is scarcely likely that these equations admit of a complete solution, but general ideas of the motion can be deduced. Supposing for an instant there were no vorticity, but only two columns of fluid of varying density, we should have $m_1=m_2=\varepsilon=0$, and so

$$\frac{d\alpha'_2}{dt}=-\frac{1}{2\pi}\frac{b}{c^2}\frac{d\sigma_1}{dt}=-\frac{ab}{c^2}\frac{da}{dt},$$

$$\frac{d\beta'_2}{dt}=0.$$

Thus β'_2 would be wholly unaffected, while α'_2 would alter from its original value ${}_0\alpha'_2$ according to the law

$$\alpha'_2={}_0\alpha'_2-\int_0^t\frac{ab}{c^2}\frac{da}{dt}dt\ldots\ldots\ldots\ldots\ldots..(29).$$

This shews that α'_2 would increase or decrease according as a were decreasing or increasing. Thus, if both columns were diminishing in cross-section, and so approaching, there would be a decided tendency in both cross-sections to assume

an elliptical sort of outline, the major axes coinciding with the line joining their centres. It is pretty obvious, taking into consideration the existence of vorticity, that the vibrations will become more important if the vortices are approaching, and will not take place about a truly circular form.

If the density of the vortices vary very slowly it is comparatively easy to trace the effect on the vibrations. Suppose

$$a^2 = a_0^2(1+\gamma t),\quad b^2 = b_0^2(1+\gamma' t);$$

and let $$m_1 = \pi a_0^2\omega,\quad m_2 = \pi b_0^2\omega'.$$

It what follows we suppose γt and $\gamma' t$ to remain small during the time considered. Now, when the fluid is incompressible, α'_2 and β'_2 experience, as is shewn by Prof. Thomson,* two forms of vibration, the shorter period being $2\pi/\omega'$ and the longer π/n, where

$$n \equiv (\omega a_0^2 + \omega' b_0^2)/c_0^2 \text{.....................} (30).$$

If then the period to which our equations are applicable be supposed to be at least several times greater than the longer period of vibration, we must have γc_0^2 and $\gamma' c_0^2$ small compared to $\omega a_0^2 + \omega' b_0^2$; thus terms in γ or γ' must be neglected when terms in ω or ω' occur. Terms in $(\gamma t)^2$ or $(\gamma' t)^2$ are negligible, and terms in n are small compared to terms in ω or ω'.

Removing the terms which are negligible according to the above hypothesis, we get for the equations in α'_2 and β'_2, from (27) and (28),

$$\left.\begin{aligned}\frac{d\alpha'_2}{dt} + \omega'(1-\gamma' t)\beta'_2 - \omega a_0^2 b_0 c^{-2}(1+\tfrac{1}{2}\gamma' t)\sin 2nt = 0\\ \frac{d\beta'_2}{dt} - \omega'(1-\gamma' t)\alpha'_2 + \omega a_0^2 b_0 c_0^{-2}(1+\tfrac{1}{2}\gamma' t)\cos 2nt = 0\end{aligned}\right\}\text{...} (31).$$

These give

$$\frac{d^2\alpha'_2}{dt^2} + \omega'^2(1-2\gamma' t)\alpha'_2 = a_0^2 b_0 \omega c_0^{-2}(\omega' + 2n - \tfrac{1}{2}\omega'\gamma' t)\cos 2nt,$$

$$\frac{d^2\beta'_2}{dt^2} + \omega'^2(1-2\gamma' t)\beta'_2 = a_0^2 b_0 \omega c_0^{-2}(\omega' + 2n - \tfrac{1}{2}\omega'\gamma' t)\sin 2nt.$$

* *Motion of Vortex Rings*, p. 77.

Suppose $\alpha'_2 = 0 = \beta'_2$ when $t = 0$, then to the same degree of approximation the solutions of these equations are

$$\alpha'_2 = \frac{a_0^2 b_0 \omega}{c_0^2(\omega' - 2n)} \left\{(1 + \tfrac{3}{2}\gamma' t)\cos 2nt - (1 + \tfrac{1}{2}\gamma' t)\cos\omega' t \right.$$
$$\left. + \tfrac{1}{2}\frac{\gamma'}{\omega'}(1 - \omega'^2 t^2)\sin\omega' t\right\} \text{......(32)},$$

$$\beta'_2 = \frac{a_0^2 b_0 \omega}{c_0^2(\omega' - 2n)} \{(1 + \tfrac{3}{2}\gamma' t)\sin 2nt - (1 + \tfrac{1}{2}\gamma' t)\sin\omega' t$$
$$+ \tfrac{1}{2}\omega'\gamma' t^2 \cos\omega' t\} \text{...............(33)}.$$

Putting $\gamma' = 0$ we obtain Prof. Thomson's results on his page 77.

The case $m_1 = -m_2$, $a = b$, which applies to a single vortex parallel to an infinite wall, has not been specially treated by Prof. Thomson. It presents certain peculiarities which seem worthy of notice. First neglecting the compressibility, the equations (30) become

$$\frac{d\alpha'_2}{dt} + \omega'\beta'_2 = 0,$$

$$\frac{d\beta'_2}{dt} - \omega'\alpha'_2 - \omega' b^2/c^2 = 0.$$

Here ω', b, c are constants; thus, if $\alpha'_2 = 0 = \beta'_2$, when $t = 0$, the solutions are

$$\left.\begin{aligned} \alpha'_2 &= -b^2 c^{-2}(1 - \cos\omega' t) \\ \beta'_2 &= b^2 c^{-2}\sin\omega' t \end{aligned}\right\} \text{...............(34)}.$$

The form of the cross-section at time t is thus given by

$$R' = b - b^3 c^{-2}\cos 2\theta' + b^3 c^{-2}\cos(2\theta' - \omega' t) \text{......(35)},$$

the wall being perpendicular to the line $\theta' = 0$ at the distance $\frac{1}{2}c$. Thus the diameter of the vortex perpendicular to the wall suffers a shortening $2b^3c^{-2}$, and that parallel to the wall an equal lengthening, and the vortex has a single disturbance, of period $2\pi/\omega'$, about this altered position.

In the same case, considering the compressibility alone, we have $\beta'_2 = \text{constant}, = 0$ say, and from (29)

$$\alpha'_2 = {}_0\alpha'_2 - \int_0^t \frac{b^2 \frac{db}{dt}\,dt}{c_0^2 + 2(b^2 - b_0^2)}.$$

Thus $$\alpha'_2 = {}_0\alpha'_2 - \tfrac{1}{3}\,\frac{b^3 - b_0^3}{c_0^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(36),$$

neglecting higher powers of $(b/c_0)^2$.

The diameter perpendicular to the wall would thus become

$$2(b+\alpha'_2) \equiv 2b + 2\,{}_0\alpha'_2 - \tfrac{2}{3}\,\frac{b^3 - b_0^3}{c_0^2},$$

and that parallel to the wall would become

$$2(b-\alpha'_2) \equiv 2b - 2\,{}_0\alpha'_2 + \tfrac{2}{3}\,\frac{b^3 - b_0^3}{c_0^2}.$$

Suppose ${}_0\alpha'_2 = 0$, then the diameter perpendicular to the wall is obviously the greater if b be decreasing, *i.e.* if the vortex be approaching the wall; the reverse is the case if the vortex be receding from the wall.

Considering then either the vorticity alone or the compressibility alone, we come to the conclusion that a single vortex in presence of an infinite wall will not retain a truly circular cross-section, and that the deviation from the circular form varies inversely as the square of the distance from the wall. When both vorticity and compressibility are considered, the deviation from the circular form will still vary inversely as the square of the distance from the wall, but the exact change in the form of the cross-section could only be deduced from a complete solution of the equations (27) and (28) for the case $\varepsilon = 0$, and this I have been unable to obtain.

DEPRESSION OF DIFFERENTIAL EQUATIONS.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's Coll., Lond., &c.

[References to Boole's *Treatise on Differential Equations*, 2nd. Ed.]

1. *General Notation.* As follows:—

x, y are always the variables of the original equation.

t, u ,, the variables of a "depressed" equation.

(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) are the variables of the first, second, third, and fourth depressed equations formed in succession.

Lagrangian notation is used for differential coefficients where compactness is required; thus:—

$y', y'', \ldots y^{(r)}$ denote differentiation with respect to					x,
$u', u'', \ldots u^{(r)}$	"	"	"	"	t,
$y_1^{(r)}, y_2^{(r)}, y_3^{(r)}, y_4^{(r)}$	"	"	"		x_1, x_2, x_3, x_4.

2. *Boole's Depressions.* It is shown in Boole's "Differential Equations," Chap. X. that a differential equation admits of depression *by one order* in each of the following cases:—

i. When not involving y.

ii. When not involving x.

iii. When homogeneous in order 1,
and when homogeneous in order m.

iv. When homogeneous in order ∞.

But it is not shown whether these depressions can be either *carried out in succession*, or *repeated.*

3. *Object of Paper.* It is proposed here to investigate the possibility of *successive* and of *repeated* depressions, thus greatly extending the power of the process of depression.

As to *successive* depression the following important result will be shown—

> "A differential equation admits of *successive* depressions as follows:—
>
> i. By r orders when the r quantities $y, y', y'', \ldots y^{(r-1)}$ are *all* absent.
>
> ii. By one order when x is absent.
>
> iii. By one order when homogeneous in *any* one order.
>
> iv. By one order when homogeneous in *any other* order."

Thus an equation not involving $x, y, y', y'', \ldots y^{(r-1)}$, and also homogeneous in two orders, admits of depression by $(r+3)$ orders.

To prove the above it will suffice to show that the successive depressions—if applied in suitable order—do not affect the remaining singularities, *i.e.* produce depressed equations still possessing those singularities.

It will also be shown that in certain cases one of the depressions can be *repeated* without affecting any of the other singularities; and that in certain cases the application of the depressions ii., iii., or iv. produces an equation possessing one of the singularities i., ii. not present in the original, and therefore susceptible of further depression.

4. *Homogeneity.* It is convenient first to investigate some properties of homogeneity.

Supposing x, y to be quantities of degrees λ, μ respectively, then the following scale of degrees obtains:—

$$\begin{array}{lllllll} \text{Quantities}\ldots x, & y, & y', & y'', & y''', & \ldots y^{(r)}, \\ \text{Degrees}\ldots\ldots\lambda, & \mu, & \mu-\lambda, & \mu-2\lambda, & \mu-3\lambda, & \ldots\mu-r\lambda. \end{array}$$

Also the *degree* (N) of any "term" Q consisting of the product of several of these quantities is clearly equal to the *sum of the degrees* of its component factors; thus, if

$$Q = a.x^p.y^q\,(y')^\alpha\,(y'')^\beta\ldots(y^{(r)})^\rho,$$

then

$$\begin{aligned} N &= p\lambda + q\mu + \alpha\,(\mu-\lambda) + \beta\,(\mu-2\lambda) + \ldots\rho\,(\mu-r\lambda) \\ &= L.\lambda + M.\mu, \end{aligned}$$

where, for shortness,

$$\begin{aligned} L &= p - (\alpha + 2\beta + 3\gamma + \ldots r\rho), \\ M &= q + (\alpha + \beta + \gamma + \ldots.\rho). \end{aligned}$$

Now, it is obvious—from first principles—that all "terms" with coefficients of equal degree connected by the signs $+$, $-$, $=$ in any sort of "equation" must necessarily be *of same degree* (N) throughout the equation; hence the theorem:—

> "Every differential equation with coefficients of equal degree in each term is homogeneous in x, y, y', y'', &c., and may therefore be depressed one order."

A homogeneous differential equation being then a sum of terms of form Q of equal degree N may be written

$$\Sigma\,(Q) = 0,$$

and the homogeneity is expressed by the following, which may be called the "equation of homogeneity,"

$$L.\lambda + M.\mu = N \text{ (a constant for every term).}$$

Def. The ratio $\nu = \mu : \lambda$ of the degrees of y and x is called the "order of homogeneity," and the quantity N is called the "degree of homogeneity."

The equation of homogeneity is of course satisfied by the values of λ, μ which express the real degrees of x, y in the primitive; it is also satisfied by any equi-multiples, say $k\lambda$, $k\mu$, as these change alike the "degree" of *every* term to kN, without altering the "order" $\nu=\mu:\lambda$. It follows that in these questions the ratio of μ, λ (here called the "order of homogeneity") is of more importance than the actual values of μ, λ: in particular changing λ, μ into $-\lambda$, $-\mu$ changes the degree into $-N$ without altering the "order."

The most useful values to ascribe to λ, μ are $\lambda=-1, 0, 1$; $\mu=-1, 0, 1, 2, \infty$, whence the following scheme:—

Order of homogeneity ν	Value of λ	Value of μ	Degrees of x	y	y'	y''	y'''	$y^{(r)}$
$\mu:\lambda$	λ	μ	λ	μ	$\mu-\lambda$	$\mu-2\lambda$	$\mu-3\lambda$	$\mu-r\lambda$
-1	-1	1	-1	1	2	3	4	$1+r$
	1	-1	1	-1	-2	-3	-4	$-1-r$
0	-1	0	-1	0	1	2	3	r
	1	0	1	0	-1	-2	-3	$-r$
1	-1	-1	-1	-1	0	1	2	$-1+r$
	λ	λ	λ	λ	0	$-\lambda$	-2λ	$(1-r)\lambda$
	1	1	1	1	0	-1	-2	$1-r$
2	-1	-2	-1	-2	1	0	-1	$-2+r$
	1	2	1	2	1	0	-1	$2-r$
∞	0	± 1	0	± 1	± 1	± 1	± 1	± 1
	0	$\pm\mu$	0	$\pm\mu$	$\pm\mu$	$\pm\mu$	$\pm\mu$	$\pm\mu$
	λ	∞	λ	∞	∞	∞	∞	∞

5. *Multiple homogeneity.* The following important proposition will now be proved:—

> "A differential equation which is homogeneous in two *different* orders ν_1, ν_2 is also homogeneous in all orders."

For, by hypothesis,

$$L.\lambda_1+M.\mu_1=N_1, \text{ (a constant for every term),}$$

$$L.\lambda_2+M.\mu_2=N_2, \text{ (a constant for every term).}$$

But these give, introducing an arbitrary multiplier k,

$$L(\lambda_1 + k\lambda_2) + M(\mu_1 + k\mu_2) = (N_1 + kN_2), \text{ also a constant,}$$

and this result exyresses that the equation has homogeneity of order $\nu_k = (\mu_1 + k\mu_2) : (\lambda_1 + k\lambda_2)$, and therefore of every order, since k is arbitrary. Q. E. D.

As to the possibility of such multiple homogeneity, the necessary and sufficient conditions are easily seen to be

$$L \text{ and } M, \text{ both constant for every term,}$$

or, writing out,

$$p - (\alpha + 2\beta + 3\gamma + \ldots + r\rho) = L, \text{ (a constant for every term),}$$

$$q + (\alpha + \beta + \gamma + \ldots + \rho) = M, \text{ (a constant for every term),}$$

and these conditions can be satisfied by more than one distinct system of values of $p, q, \alpha, \beta, \ldots \rho$ only when at least three of these quantities enter into every pair of systems, or (which is the same thing) when at least three of the quantities $x, y, y', \ldots y^{(r)}$ appear in every pair of terms: hence

> "Multiple homogeneity is possible only in an equation containing at least three of the quantities $x, y, y', \ldots y^{(r)}$ in every pair of terms."

The fact that duplex homogeneity involves multiple homogeneity bears the following consequences:—

> "A differential equation which has multiple homogeneity cannot in general be depressed more than two orders in virtue solely of homogeneity."

> "The depression-formulæ suited to any order of homogeneity that is convenient may be applied to depress a differential equation which has homogeneity of *any* two orders."

Examples of multiple homogeneity. It is worth notice that multiple homogeneity occurs* in all final differential equations of rational algebraic equations (formed by elimination of all the constants).

6. Depression i. *Absence of* $y, y', y'', \ldots y^{(r-1)}$ (r quantities).

The depression-formulæ are (*Boole*, Ch. X., Art. 1)

$$x = t, \quad y^{(r)} = u, \quad y^{(r+1)} = u', \quad y^{(r+2)} = u'', \quad \ldots y^{(n)} = u^{(n-r)}.$$

---

* This property will be proved in another paper.

Thus this depression introduces the new independent variable (t) only in place of the old one (x) when present in the original.

As to the effect on homogeneity, it is clear that if x, y be reckoned of degrees λ, μ, then t, u must be reckoned like x, $y^{(r)}$, *i.e.* of degrees λ, $\mu - r\lambda$ (Art. 4). Thus x, $y^{(r)}$, $y^{(r+1)}$, ...$y^{(n)}$ are replaced by the quantities t, u, u', ...$u^{(n-r)}$ of degrees λ, M, $(M-\lambda)$, $(M-2\lambda)$, ...$\{M-(n-r)\lambda\}$, writing $M=(\mu - r\lambda)$ for shortness; *i.e.* by quantities of like degree with themselves. Thus the *degree* of each term in the depressed equation is the same as that of the corresponding term in the original. Hence, if the original had single homogeneity of any *order* $\mu : \lambda$ and *degree* N, the depressed equation has single homogeneity of order $M : \lambda$ and of same degree N. Similarly if the original has multiple homogeneity, so also has the depressed equation.

Thus this depression does not destroy the remaining singularities ii., iii., iv.; but it introduces the new dependent variable (u), so cannot be repeated (unless $y^{(r)}$, $y^{(r-1)}$, &c. be also wanting). As this depression is much easier of application than depression ii., iii., iv., it would usually be applied so as to depress *by as many orders as possible* at one step, and therefore could not be repeated, because $y^{(r)}$ is hereby supposed present and would introduce u.

7. DEPRESSION ii. *Absence of x.* The depression-formulæ are (*Boole*, CH. X., Art. 1),

$$y=t, \quad y'=u, \quad y''=uu', \quad y'''=u^2u''+uu'^2,$$
$$y^{iv}=u^3u'''+4u^2u'u''+uu'^3, \text{ \&c.};$$

and, in general, $\qquad D_x^{r+1}y=(u\,.\,D_t)^r\,u.$

As these depression-formulæ thus generally introduce the new dependent variable (u), with one exception noted hereafter (Art. 21, wherein u cancels out) depression i. cannot be applied after the present (except in the case reserved). Also this depression always introduces the new independent variable (t) in place of y, and therefore cannot be repeated when y was either originally present or when introduced by depression i, (see remarks at end of last Article).

As to the effect on homogeneity, the depression-formulæ shew that if x, y be reckoned of degrees λ, μ, then t, u must be reckoned like y, y' of degrees μ, $\mu-\lambda$ respectively,

so that

the quantities

$$t, \quad u, \quad u', \quad u'', \quad u''', \quad \ldots \quad u^{(r)}$$

are of degrees

$$\mu, \ (\mu-\lambda), \ -\lambda, \ -\lambda-\mu, \ -\lambda-2\mu, \ \ldots -\lambda-(r-1)\mu.$$

On referring to the depression-formulæ, it will be seen that $y, y', y'', \ldots y^{(r)}$ are replaced by homogeneous functions of $t, u, u', \ldots u^{(r-1)}$ of same degrees as their own. Hence, if the original equation have homogeneity of any order $\mu : \lambda$ and of degree N (so that each of its terms are of degree N), the depressed equation has homogeneity of order $(\mu-\lambda) : \mu$ (the ratio of the degrees of u, t), and of the same degree (N) as the original. Hence also, if the original equation has multiple homogeneity, the depressed will also have multiple homogeneity.

8. Depression iii. *Homogeneity of order* ν. Here ν may be zero, but not infinite. The depression-formulæ are (*Boole*, Ch. X., Art. 3, Class II)

$$x = \varepsilon^{\int\frac{dt}{u}}, \quad y = x^{\nu}.t,$$

so that $$D_x^{r+1}y = (\varepsilon^{-\int\frac{dt}{u}}.uD_t)^r \{\varepsilon^{(\nu-1)\int\frac{dt}{u}}(u+\nu t)\}.$$

The quantity x or $\varepsilon^{\int\frac{dt}{u}}$ will be found to cancel out of the depressed equation, which may be formed directly by replacing

$$x \text{ by } 1, \quad y \text{ by } t, \quad y' \text{ by } (u+\nu t),$$
$$y'' \text{ by } \{uu' + (2\nu-1)u + \nu(\nu-1)t\},$$
$$y''' \text{ by } \{u^2u'' + uu'^2 + 3(\nu-1)uu' + (3\nu^2-6\nu+2)u$$
$$+\nu(\nu-1)(\nu-2)t\}, \quad \&c.$$

As these depression-formulæ thus generally involve both of the new variables (t, u) with certain exceptions considered hereafter (Arts. 21, 22) (wherein t, u cancel out) depressions i., ii. cannot be applied after the present (except in the cases reserved).

As to the effect on homogeneity, the depression-formulæ give

$$t = x^{-\nu}.y, \quad u = x.D_xt,$$

from which it follows that, if x, y be reckoned of degrees λ, μ; then (since $\nu=\mu:\lambda$) t, u must each be reckoned of equal zero degree, or in other words of equal infinitesimal degree i, so that, by Art. 4,

$$\begin{array}{lcccccc} \text{the quantities} & t, & u, & u', & u'', & u''', & \ldots \quad u^{(r)} \\ \text{are of degrees} & i, & i, & 0, & -i, & -2i, & \ldots -(r-1)\,i. \end{array}$$

On referring to the depression-formulæ, it will be seen that x is replaced by a quantity of zero degree, whilst the rest of the quantities y, y', y'', &c. are replaced by homogeneous functions of t, u, u', u'', &c. all of equal degree i.

Hence this depression destroys the homogeneity of an equation possessing *single* homogeneity of order ν (not infinite), and therefore cannot be repeated in equations of single homogeneity.

But an equation possessing multiple homogeneity has necessarily homogeneity of order ∞ (by Art. 5), wherein x is reckoned of degree zero, and y, y', y'', &c. all of equal degree. But these have been shewn above to be the degrees of the functions (of t, u, u', u'', &c.) which replace them. Hence, in cases of original multiple homogeneity, this depression produces an equation with *single* homogeneity of same *degree* (N) as the original, and of first *order* (since t, u are equal degree).

Hence an equation possessing multiple homogeneity admits of two successive depressions (and no more) in virtue thereof, viz.

1st. One depression by the formulæ for homogeneity of any arbitrary (not infinite) order (ν): this leaves single homogeneity of first order.

2nd. One more depression by formulæ for homogeneity of first order: this destroys the homogeneity.

9. DEPRESSION iv. *Homogeneity of order* ∞. The depression-formulæ are (*Boole*, CH. X., Art. 3, Class III)

$$x=t,\ y=\varepsilon^{\int u\,dt},\ \text{so that}\ D_x^{\,r+1}y=D_t^{\,r}(\varepsilon^{\int u\,dt}.u),$$

and the quantity y or $\varepsilon^{\int u\,dt}$ is found to cancel out of the depressed equation, which may be formed directly by replacing

$$x \text{ by } t,\quad y \text{ by } 1,\quad y' \text{ by } u,\quad y'' \text{ by } (u'+u^2),$$
$$y''' \text{ by } (u''+3uu'+u^3),$$
$$y^{\text{iv}} \text{ by } (u'''+4uu''+3u'^2+6u^2u'+u^4),\ \text{\&c.}$$

As these depression-formulæ thus generally introduce the new dependent variable (u), with an exception noted hereafter (Art. 21) (wherein u cancels out), depression i. cannot be applied after the present (except in the case reserved).

Again, as they introduce the new independent variable (t) only in place of the old one (x), this depression does not affect the applicability of depression ii., which can therefore be applied after the present if x was originally absent.

As to the effect on homogeneity, the depression-formulæ give

$$t=x, \quad u=\frac{1}{y}D_x y,$$

from which it follows that, if x be reckoned as of degree -1, and y, y', y'', &c. all of equal infinite degree (see Art. 4), then t must be reckoned (like x) of degree -1, and u of degree 1, so that (by Art. 4)

the quantities	t,	u,	u',	u'',	u''',	...	$u^{(r)}$
are of degrees	-1,	1,	2,	3,	4,	...	$(r+1)$.

On referring to the depression-formulæ it will be seen that x, y, y', y'', &c. are replaced by homogeneous functions of t, u, u', u'', &c. of degrees as below:—

Degree of function	-1,	0,	1,	2,	3,	...	r,
replacing	x,	y,	y',	y'',	y''',	...	$y^{(r+1)}$.

Hence this depression destroys the homogeneity of an equation possessing single homogeneity of order ∞, and therefore cannot be repeated on such an equation. But an equation possessing multiple homogeneity has necessarily homogeneity of order zero (Art. 5), wherein x is reckoned of degree -1, and y, y', y'', &c. of degrees 0, 1, 2, &c. But these have been shewn above to be the degrees of the functions (of t, u, u', &c.) which replace them. Hence, in cases of original multiple homogeneity, this depression produces an equation with *single* homogeneity of same *degree* (N) as the original and of *order* -1 (since t, u are of degrees -1, 1).

Hence an equation possessing multiple homogeneity admits of two successive depressions (and no more) in virtue thereof, viz.

1st. One depression by formulæ for homogeneity of order ∞: this leaves single homogeneity of order -1.

2nd. One more depression by formulæ for homogeneity of order -1: this destroys the homogeneity.

10. *Depression-Formulæ.* Table II. shews the functions of t, u, u', u'', u''' that are to be substituted for x, y, y', y'', y^{iv} in performing any one depression, and also the general substitution for $y^{(r)}$. The substitutions for homogeneity are *not the actual values* of x, y, y', &c., the functions of x, y which would necessarily cancel out of the depressed equation being omitted.

11. *Elevation-Formulæ.* Table III. shews the functions of x, y, y', y'', &c. that are to be substituted for t, u, u', u'', &c. in raising the order of any differential equation in t, u, also the general substitution for $u^{(m)}$. These substitutions are actual equivalencies.

Depression-Formulæ.

Table II.

[x, y the original variables; t, u the variables in depressed equation].

N.B. In these Tables the sign $\equiv$ should be read as *becomes*, or *may be changed into*.

Depression i. $y, y', y'', \ldots y^{(r-1)}$ *wanting*.

$$x = t, \quad y^{(r)} = u, \quad y^{(r+1)} = u', \quad y^{(r+2)} = u'', \text{ \&c.}, \quad y^{(r+m)} = u^{(m)}.$$

Depression ii. x *absent*.

$$y = t, \quad y' = u, \quad y'' = uu', \quad y''' = u^2u'' + uu'^2,$$

$$y^{iv} = u^3u''' + 4u^2u'u'' + uu'^3,$$

$$y^{(r)} = (uD_t)^{r-1} u.$$

Depression iii. *Homogeneity of order* $v = -1$.

$$x \equiv 1, \quad y \equiv t, \quad y' \equiv u - t, \quad y'' \equiv uu' - 3u + 2t,$$

$$y''' \equiv u^2u'' + uu'^2 - 6uu' + 11u - 6t,$$

$$y^{iv} \equiv u^3u''' + 4u^2u'u'' + uu'^3 - 10\,(u^2u'' + uu'^2) + 35uu' - 50u + 24t,$$

$$y^{(r)} \equiv (\varepsilon^{-\int \frac{dt}{u}} uD_t)^{r-1} \{\varepsilon^{-2\int \frac{dt}{u}} (u - t)\}.$$

Depression iii. *Homogeneity of order* $\nu = 0$.

$$x \equiv 1, \quad y \equiv t, \quad y' \equiv u, \quad y'' \equiv uu' - u,$$
$$y''' \equiv u^2u'' + uu'^2 - 3uu' + 2u,$$
$$y^{iv} \equiv u^3u''' + 4u^2u'u'' + uu'^3 - 6(u^2u'' + uu'^2) + 11uu' - 6u.$$
$$y^{(r)} \equiv \left(\varepsilon^{-\int\frac{dt}{u}} uD_t\right)^{r-1} \left\{\varepsilon^{-\int\frac{dt}{u}} u\right\}.$$

Depression iii. *Homogeneity of order* $\nu = 1$.

$$x \equiv 1, \quad y \equiv t, \quad y' \equiv u + t, \quad y'' \equiv uu' + u,$$
$$y''' \equiv u^2u'' + uu'^2 - u,$$
$$y^{iv} \equiv u^3u''' + 4u^2u'u'' + uu'^3 - 2(u^2u'' + uu'^2) - uu' + 2u,$$
$$y^{(r)} \equiv \left(\varepsilon^{-\int\frac{dt}{u}} uD_t\right)^{r-1} (u + t).$$

Depression iii. *Homogeneity of order* $\nu = 2$.

$$x \equiv 1, \quad y \equiv t, \quad y' \equiv u + 2t, \quad y'' \equiv uu' + 3u + 2t,$$
$$y''' \equiv u^2u'' + uu'^2 + 3uu' + 2u,$$
$$y^{iv} \equiv u^3u''' + 4u^2u'u'' + uu'^3 + 2(u^2u'' + uu'^2) - uu' - 2u,$$
$$y^{(r)} \equiv \left(\varepsilon^{-\int\frac{dt}{u}} uD_t\right)^{r-1} \left\{\varepsilon^{\int\frac{dt}{u}} (u + 2t)\right\}.$$

Depression iii. *Homogeneity of order* ν.

$$x \equiv 1, \quad y \equiv t, \quad y' \equiv u + \nu t, \quad y'' \equiv uu' + (2\nu - 1)u + \nu(\nu - 1)t,$$
$$y''' \equiv u^2u'' + uu'^2 + 3(\nu - 1)uu' + (3\nu^2 - 6\nu + 2)u + \nu(\nu - 1)(\nu - 2)t,$$
$$y^{iv} \equiv u^3u''' + 4u^2u'u'' + uu'^3 + (4\nu - 6)(u^2u'' + uu'^2) + (6\nu^2 - 18\nu + 11)uu'$$
$$+ (4\nu^3 - 18\nu^2 + 22\nu - 6)u + \nu(\nu - 1)(\nu - 2)(\nu - 3)t,$$
$$y^{(r)} \equiv \left(\varepsilon^{-\int\frac{dt}{u}} uD_t\right)^{r-1} \left\{\varepsilon^{(\nu-1)\int\frac{dt}{u}} (u + \nu t)\right\}.$$

Depression iv. *Homogeneity of order* ∞.

$$x \equiv t, \quad y \equiv 1, \quad y' \equiv u, \quad y'' \equiv u' + u^2,$$
$$y''' \equiv u'' + 3uu' + u^3,$$
$$y^{iv} \equiv u''' + 4uu'' + 3u'^2 + 6u^2u' + u^4,$$
$$y^{(r)} \equiv D_t^{(r-1)} \left\{\varepsilon^{\int u\,dt} u\right\}.$$

CONTENTS.

The following papers have been received:

Mr. Buchheim, "Note on Matrices in Involution."

Prof. Cayley, "Analytical formulæ in regard to an octad of points," "A correspondence of confocal Cartesians with the right lines of a hyperboloid,' "Note on the relation between the distances of five points in space."

Prof. Mathews, "Geometry on a quadric surface."

Articles for insertion will be received by the Editor, or by Messrs. Metcalfe and Son, Printing Office, Trinity Street, Cambridge.

No. CC.] NEW SERIES. [December, 1887.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XVII.—NO. 8.

MACMILLAN AND CO.

London and Cambridge.

1887.

Price—One Shilling.

METCALFE AND SON, CAMBRIDGE.

ELEVATION-FORMULÆ.

TABLE III.

[t, u the original variables; x, y the variables of the raised equation].

ELEVATION i. $y, y', y'', \ldots y^{(r-1)}$ *wanting.*

$$t=x,\ u=y^{(r)},\ u'=y^{(r+1)},\ u''=y^{(r+2)},\ \&c.,\ u^{(m)}=y^{(r+m)}.$$

ELEVATION ii. *x absent.*

$$t=y,\ u=y',\ u'=\frac{y''}{y'},\ u''=\frac{y'y'''-y''^2}{y'^3},$$

$$u'''=\frac{y'^2y^{\text{iv}}-4y'y''y'''+3y''^3}{y'^5},$$

$$u^{(m)}=\left(\frac{1}{y'}D_x\right)^m y'.$$

ELEVATION iii. *Homogeneity of order* $\nu=0$

$$t=y,\ u=xy',\ u'=\frac{xy''}{y'}+1,$$

$$u''=\frac{y'(xy'''+y'')-xy''^2}{y'^3},$$

$$u'''=\frac{xy'(y'y^{\text{iv}}-2y''y''')+(2y'y'''-3y''^2)(y'-xy'')}{y'^5},$$

$$u^{(m)}=\left(\frac{1}{y'}D_x\right)^m xy'.$$

ELEVATION iii. *Homogeneity of order* $\nu=1$.

$$t=\frac{y}{x},\ u=y'-\frac{y}{x},\ u'=\frac{x^2y''}{xy'-y}-1,$$

$$u''=\frac{x^3(xy'''+2y'')}{(xy'-y)^2}-\frac{x^5y''^2}{(xy'-y)^3},$$

$$u^{(m)}=\left(\frac{x^2}{xy'-y}D_x\right)^m\frac{xy'-y}{x}.$$

ELEVATION iii. *Homogeneity of order ν.*

$$t=\frac{y}{x^{\nu}},\ u=\frac{xy'-\nu y}{x^{\nu}},\ u'=\frac{x^2y''-(\nu-1)xy'}{xy'-\nu y}-\nu,$$

$$u''=\left\{\frac{x^2y'''-(\nu-3)xy''-(\nu-1)y'}{(xy'-\nu y)^2}-\frac{x\{xy''-(\nu-1)y'\}^2}{(xy'-\nu y)^3}\right\}x^{\nu+1},$$

$$u^{(m)}=\left(\frac{x^{\nu+1}}{xy'-\nu y}D_x\right)^m\frac{xy'-\nu y}{x^{\nu}}.$$

ELEVATION iv. *Homogeneity of order ∞.*

$$t=x,\ u=\frac{y'}{y},\ u'=\frac{yy''-y'^2}{y^2},$$

$$u''=\frac{y^2y'''-3yy'y''+2y'^3}{y^3},$$

$$u'''=\frac{yy^{iv}-4y'y'''-3y''^2}{y^2}+\frac{6y'^2}{y^4}(2yy''-y'^2),$$

$$u^{(m)}=D_x^{(m)}\left(\frac{y'}{y}\right).$$

12. *Successive Depression.* Summing up, it is seen that to apply the four Depressions in succession; the following must be attended to:—

Depression i. This should always be applied first, as it does not affect the remaining Depressions, whereas they—if applied first—will generally prevent its application (by introducing u).

Depression ii. This should precede Depression iii. (because the latter in general introduces t); it changes the order of homogeneity (when single) from $\nu=\mu:\lambda$ to $\nu=\mu-\lambda;\mu$.

Depression iii. This in general introduces t, and reduces multiple homogeneity to single of order $\nu=1$, so cannot precede Depressions ii. or iv. Hence in cases of multiple homogeneity the procedure is:—

(1) *When not preceded by Depression* iv. it may be applied once with any convenient value of ν, and again with the value $\nu=1$.

(2) *When preceded immediately by Depression* iv. it must be applied with value $\nu=-1$.

(3) *When preceded by Depressions* iv. *and* ii. *in turn* it must be applied with value $\nu=2$.

Depression iv. This may precede either Depressions ii. or iii., but cannot follow Depression iii. (because the latter leaves homogeneity only of order $\nu=+1$); it reduces multiple homogeneity to single of order $\nu=-1$.

The following Table shews the order in which the Depressions should be applied to equations containing two or more of the singularities detailed.

TABLE IV.

	Two Singularities.							
Singularity No.	i., ii.	i., iii. $(\nu = m)$	i., iv.	ii., iii. $(\nu = m)$	ii., iv.		iii., iv.	
					Either		Either	
Depression Step I. ...	i.	i.	.	ii.	ii.	iv.	iii. $(\nu = \nu)$	iv.
" " II. ...	.	iii. $(\nu = m)$	iv.	iii. or iv. $\nu = \left(1 - \frac{1}{m}\right)$	iii. $(\nu = 0)$	ii.	iii. $(\nu = 1)$	iii. $(\nu = -1)$

	Three Singularities.								Four Singularities.		
Singularity No.	i., ii., iii. $(\nu = m)$	i., ii., iv.		i., iii., iv.		ii., iii., iv.			i., ii., iii., iv.		
		Either		Either		Either			Either		
Depression Step I. ...	i.	i.	i.	i.	i.	ii.	ii.	iv.	i.	i.	i.
" " II. ...	ii.	ii.	iv.	iii. $(\nu=\nu)$	iv.	iii. $(\nu=\nu)$	iv.	ii.	ii.	ii.	iv.
" " III. ...	Either iii. or iv. $\nu = \left(1 - \frac{1}{m}\right)$	iii. $(\nu=0)$	ii.	iii. $(\nu=1)$	iii. $(\nu=-1)$	iii. $(\nu=1)$	iii. $(\nu=-1)$	iii. $(\nu=2)$	iii. $(\nu=\nu)$	iv.	ii.
" " IV. ...	—	—	—	—	—	—	—	—	iii. $(\nu=1)$	iii. $(\nu=-1)$	iii. $(\nu=2)$

Thus in certain cases there are two or more Courses open, viz.

2 Courses in equations with two of the singularities Nos. ii., iii., iv.

3 Courses in equations with singularities Nos. ii., iii., iv.

The particular Course most advantageous will of course depend on the nature of the question; but it will probably generally be advantageous to use Depression iii. with the general value of ν, whenever available, as this leaves a constant to be hereafter determined in such away as to simplify the integrations.

13. *Combined Depression-Formulæ.* The results of substitution by the several Depressions applied in succession may be readily computed once for all so as to shew the result after any number of successive Depressions. These are exhibited in Tables V, VI, in which

x, y, y', y'', &c., belong to the original equation,
x_1, y_1, y_1', y_1'', &c., ,, ,, ,, first depressed equation,
x_2, y_2, y_2', y_2'', &c., ,, ,, ,, second ,, ,,
x_3, y_3, y_3', y_3'', &c., ,, ,, ,, third ,, ,,
x_4, y_4, y_4', y_4'', &c., ,, ,, ,, fourth ,, ,,

By the aid of such a Table the first, second, third, or fourth depressed equation may be formed *at once* without the labour of forming the intermediate depressions by simply substituting the functions in the Table belonging to the required depressed equation for the original x, y, y', &c.

Successive Depressions.

Table V.

Depression i. *Step I in all cases.*

$$x = x_1,\ y^{(r)} = y_1,\ y^{(r+1)} = y_1',\ y^{(r+2)} = y_1'',\ y_1^{(r+3)} = y_1''', \text{ \&c.} \dots$$

Two Successive Depressions.

Depressions i. *and* ii., *combined.*

$$y_1 = x_2,\ y_1' = y_2,\ y_1'' = y_2 y_2',\ y_1''' = y_2(y_2 y_2'' + y_2'^2),$$
$$y_1^{iv} = y_2(y_2^2 y_2''' + 4y_2 y_2' y_2'' + y_2'^3).$$

Depressions i. *and* iii. $(\nu = 0)$, *combined.*

$$x_1 \equiv 1,\ y_1 \equiv x_2,\ y_1' \equiv y_2,\ y_1'' \equiv y_2(y_2' - 1),$$
$$y_1''' \equiv y_2^2 y_2'' + y_2(y_2' - 1)(y_2' - 2),$$
$$y_1^{iv} \equiv y_2^2(y_2 y_2''' + 4y_2' y_2'' - 6y_2'') + y_2(y_2' - 1)(y_2' - 2)(y_2' - 3).$$

Depressions i. *and* iii. ($\nu = 1$), *combined.*

$$x_1 \equiv 1,\ y_1 \equiv x_2,\ y_1' \equiv y_2 + x_2,\ y_1'' \equiv y_2(y_2' + 1),$$
$$y_1''' \equiv y_2(y_2 y_2'' + y_2'^2 - 1),$$
$$y_1^{iv} \equiv y_2^2(y_2 y_2''' + 3y_2' y_2'') + y_2(y_2' - 2)(y_2 y_2'' + y_2'^2 - 1).$$

Depressions i. *and* iii. ($\nu = \nu$), *combined.*

$$x_1 \equiv 1,\ y_1 \equiv y_2,\ y_1' \equiv y_2 + \nu x_2$$
$$y_1'' \equiv y_2(y_2' + 2\nu - 1) + \nu(\nu - 1)x_2,$$
$$y_1''' \equiv y_2(y_2 y_2'' + y_2'^2) + 3(\nu - 1)y_2 y_2' + (3\nu^2 - 6\nu + 2)y_2 + \nu(\nu - 1)(\nu - 2)x_2,$$
$$y_1^{iv} \equiv y_2(y_2^2 y_2''' + 4y_2 y_2' y_2'' + y_2'^3) + (4\nu - 6)y_2(y_2 y_2'' + y_2'^2) + (6\nu^2 - 18\nu + 11)y_2 y_2' + (4\nu^3 - 18\nu^2 + 22\nu - 6)y_2 + \nu(\nu - 1)(\nu - 2)(\nu - 3)x_2.$$

Depressions i. *and* iv. *combined.*

$$x_1 \equiv x_2,\ y_1 \equiv 1,\ y_1' \equiv y_2,\ y_1'' \equiv y_2' + y_2^2,$$
$$y_1''' \equiv y_2'' + 3y_2 y_2' + y_2^3,$$
$$y_1^{iv} \equiv y_2''' + 4y_2 y_2'' + 3y_2'^2 + 6y_2^2 y_2' + y_2^4.$$

Three Successive Depressions.

Depressions i., ii., *and* iii. ($\nu = 0$), *combined.*

$$y_1 \equiv 1,\ y_1' \equiv x_3,\ y_1'' \equiv x_3 y_3,\ y_1''' \equiv x_3 y_3(x_3 y_3' + y_3 - x_3),$$
$$y_1^{iv} \equiv x_3 y_3\{x_3^2 y_3 y_3'' + x_3^2(y_3' - 1)(y_3' - 2) + 4x_3 y_3(y_3' - 1) + y_3^2\}.$$

Depressions i., ii., *and* iii. ($\nu = 1$, *combined*).

$$y_1 \equiv 1,\ y_1' \equiv x_3,\ y_1'' \equiv x_3(y_3 + x_3),$$
$$y_1''' \equiv x_3(x_3 y_3 y_3' + y_3^2 + 3x_3 y_3 + x_3^2),$$
$$y_1^{iv} \equiv x_3\{x_3^2 y_3(y_3 y_3'' + y_3'^2 + 4y_3' + 3) + 4x_3 y_3^2(y_3' + 1) + (y_3 + x_3)^3\}.$$

Depressions i., ii., *and* iii. ($\nu = \nu$) *combined.*

$$y_1 \equiv 1,\ y_1' \equiv x_3,\ y_1'' \equiv x_3(y_3 + \nu x_3),$$
$$y_1'' \equiv x_3\{x_3 y_3(y_3' + 4\nu - 1) + y_3^2 + \nu(2\nu - 1)x_3^2\},$$
$$y_1^{iv} \equiv x_3^3\{y_3^2 y_3'' + y_3 y_3'^2 + 3(\nu - 1)y_3 y_3' + (3\nu^2 - 6\nu + 2)y_3 + \nu(\nu - 1)(\nu - 2)(\nu - 3)x_3\} + 4x_3^2(y_3 + \nu x_3) \times\{y_3 y_3' + (2\nu - 1)y_3 + \nu(\nu - 1)x_3\} + x_3(y_3 + \nu x_3)^3.$$

Depressions i., ii., *and* iv., *combined.*

$$y_1 \equiv x_3,\ y_1' \equiv 1,\ y_1'' \equiv y_3,\ y_1''' \equiv y_3' + 2y_3^2,$$
$$y_1^{iv} \equiv y_3'' + 7y_3y_3' + 6y_3^3.$$

Depressions i., iii. ($\nu = 0$), *and* iii. ($\nu = 1$), *combined.*

$$x_1 \equiv 1,\ y_1 \equiv 1,\ y_1' \equiv x_3,\ y_1'' \equiv x_3(y_3 + x_3 - 1),$$
$$y_1''' \equiv x_3^2y_3(y_3' + 1) + x_3(y_3 + x_3 - 1)(y_3 + x_3 - 2),$$
$$y_1^{iv} \equiv x_3^2y_3\{x_3(y_3y_3'' + y_3'^2 - 1) + (y_3' + 1)(4y_3 + 4x_3 - 6)\}$$
$$+ x_3(y_3 + x_3 - 1)(y_3 + x_3 - 2)(y_3 + x_3 - 3).$$

Depressions i., iii. ($\nu = 1$), *and* iii. ($\nu = 1$) *combined.*

$$x_1 \equiv 1,\ y_1 \equiv 1,\ y_1' \equiv x_3 + 1,\ y_1'' \equiv x_3(y_3 + x_3 + 1),$$
$$y_1''' \equiv x_3^2y_3(y_3' + 1) + x_3\{(y_3 + x_3)^2 - 1\},$$
$$y_1^{iv} \equiv x_3^2y_3\{x_3(y_3y_3'' + y_3'^2 - 1) + 3(y_3 + x_3)(y_3' + 1)\}$$
$$+ x_3(y_3 + x_3 - 2)\{x_3y_3(y_3' + 1) + (y_3 + x_3)^2 - 1\}.$$

Degressions i., iii. ($\nu = \nu$), *and* iii. ($\nu = 1$), *combined.*

$$x_1 \equiv 1,\ y_1 \equiv 1,\ y_1' = x_3 + \nu,\ y_1'' \equiv x_3y_3 + (x_3 + \nu)(x_3 + \nu - 1),$$
$$y_1''' \equiv x_3y_3\{x_3y_3' + y_3^2 + 3(x_3 + \nu - 1)\}$$
$$+ (x_3 + \nu)(x_3 + \nu - 1)(x_3 + \nu - 2),$$
$$y_1^{iv} \equiv x_3^3y_3(y_3y_3'' + y_3'^2 + 4y_3' + 6) + x_3^2y_3\{(4y_3 + 4\nu - 6)y_3'$$
$$+ 7y_3 + 6(2\nu - 3)\} + x_3y_3\{y_3^2 + (4\nu - 6)y_3 + (6\nu^2 - 18\nu + 11)\}$$
$$+ (x_3 + \nu)(x_3 + \nu - 1)(x_3 + \nu - 2)(x_3 + \nu - 3).$$

Depressions i., iv. *and* ii. *combined.*

$$y_1 \equiv 1,\ y_1' \equiv x_3,\ y_1'' \equiv y_3 + x_3^2,$$
$$y_1''' \equiv y_3y_3' + 3x_3y_3 + x_3^3,$$
$$y_1^{iv} \equiv y_3(y_3y_3'' + y_3'^2 + 4x_3y_3' + 6x_3^2 + 3y_3) + x_3^4.$$

Depressions i., iv. *and* iii. ($\nu = -1$) *combined.*

$$x_1 \equiv 1,\ y_1 \equiv 1,\ y_1' \equiv x_3,\ y_1'' \equiv y_3 + x_3(x_3 - 1),$$
$$y_1''' \equiv y_3(y_3' + 3x_3 - 3) + x_3(x_3 - 1)(x_3 - 2),$$
$$y_1^{iv} \equiv y_3\{y_3y_3'' + y_3'^2 + 4x_3y_3' - 6y_3' + 3y_3 - 1 + 6(x_3 - 1)(x_3 - 2)\}$$
$$+ x_3(x_3 - 1)(x_3 - 2)(x_3 - 3).$$

Four Successive Depressions.

Depressions i., ii., iii. ($\nu=0$), *and* iii. ($\nu=1$), *combined.*

$$y_1\equiv 1,\ y_1'\equiv 1,\ y_1''\equiv x_4,\ y_1'''\equiv x_4(y_4+2x_4-1),$$
$$y_1^{iv}\equiv x_4\{x_4y_4y_4'+y_4^2+7x_4(y_4-1)+6x_4^2-3y_4+2\}.$$

Depressions i., ii., iii. ($\nu=1$), *and* iii. ($\nu=1$), *combined,*

$$y_1\equiv 1,\ y_1'\equiv 1,\ y_1''\equiv x_4+1,$$
$$y_1'''\equiv x_4y_4+(x_4+1)(2x_4+1),$$
$$y_1^{iv}\equiv x_4^2(y_4y_4'+y_4)+x_4(y_4+x_4+1)(y_4+5x_4+3)+(x_4+1)^3.$$

Depressions i., ii., iii. ($\nu=\nu$), *and* iii. ($\nu=1$), *combined.*

$$y_1\equiv 1,\ y_1'\equiv 1,\ y_1''\equiv x_4+\nu,$$
$$y_1'''\equiv x_4y_4+(x_4+\nu)(2x_4+2\nu-1),$$
$$y_1^{iv}\equiv x_4y_4(x_4y_4'+y_4+7x_4+7\nu-3)$$
$$+(x_4+\nu)(3x_4+3\nu-2)(2x_4+2\nu-1).$$

Depressions i., ii., iv. *and* iii. ($\nu=-1$), *combined.*

$$y_1\equiv 1,\ y_1'\equiv 1,\ y_1''\equiv x_4,\ y_1'''\equiv y_4-x_4+2x_4^2,$$
$$y_1^{iv}\equiv y_4(y_1'+7x_4-3)+x_4(2x_4-1)(3x_4-2).$$

Depressions i., iv., ii. *and* iii. ($\nu=2$), *combined.*

$$y_1\equiv 1,\ y_1'\equiv 1,\ y_1''\equiv x_4+1,$$
$$y_1'''\equiv x_4y_4+(x_4+1)(2x_4+1),$$
$$y_1^{iv}\equiv x_4^2(y_4y_4'+7y_4+17x_4+6)+1.$$

14. *Repeated Depression.* The several depressions considered above may in some cases be repeated.

15. *Depression* i. *repeated.* The substitution $x=t$, $y^{(r)}=u$, applicable when the r terms $y, y', y'', \ldots, y^{(r-1)}$ are all wanting, and which depresses r orders at one step, is evidently equivalent to r repetitions of this depression, depressing one order at each step, viz. by the substitutions

$$(x=x_1, y'=y_1),\ (x_1=x_2, y_1'=y_2),\ \ldots,\ (x_{r-1}=t, y'_{r-1}=u),$$

but, as the same final result may also be obtained by the single substitution ($x=t$, $y^{(r)}=u$), there is no advantage in this repetition.

16. *Depression* ii. *repeated.* On referring to the depression-formulæ (Art. 7), it is seen that (in the absence of x) y, y'

become t, u, and t does not recur in the substitutions for y'', y''', &c. Hence, if y be absent from the original, t will be absent from the depressed equation, so that depression ii. may be applied a second time.

The *à priori* recognition of the further applicability of this depression is more difficult. Thus, if u be absent from the first depressed equation (as well as t), this depression could be applied again. Inasmuch as $y' = u$, and, as further, u enters into the substitutions for y'' and all higher differential coefficients (Art. 7), it is clear that the absence of u is secured only when y', y'', y''', &c. occur in the forms (given in Table III) equivalent to u', u'', &c., (and in no other).

The results of the double application of this depression, applicable when x, y are both absent from the original equation, are shown in Table VII, with the new variables of Art. 13. It is probable that this double application will not often be advantageous, as the depression by two orders might in the same case (x, y both absent) be generally *more simply* effected by the use of depressions i., ii. applied in succession.

The results of further application of this depression cannot be shown in an equally compendious way with the last.

As by Art. 7, a single application of depression ii. does not destroy homogeneity (merely changing the order, when single), it follows easily that repeated application will not destroy homogeneity, merely changing its order if single.

17. *Depression* iii. *repeated.* The *à priori* recognition of repeated applicability (with same value of ν) is by no means simple. The original equation must of course possess homogeneity of some finite order (ν) in x, y. Besides which the depressed equation must also have homogeneity of the same finite order (ν) in t, u; this involves that the original equation should contain x, y, y', y'', &c., only in the forms given in Table III as the equivalents of t, u, u', u'', &c., and should possess homogeneity of order ν in those functions; a condition so complex as to make the *à priori* recognition of the repeated applicability of this depression difficult. There is, however, one case of easy recognition, viz. that of multiple homogeneity, in which it has been shown (Art. 8) that this depression may be effected twice with the value $\nu = 1$ in each instance.

By reasoning similar to that in Art. 8, it is seen that the repeated application of this depression generally destroys the applicability of depression ii. (by introducing the independent variable); also, that in an equation possessing

multiple homogeneity as well as the property above mentioned, one application of this depression will still leave multiple homogeneity.

18. *Depression* iv. *repeated.* The *à priori* recognition of repeated applicability is by no means simple. The original equation must of course have homogeneity of order ∞ in x, y; and the depressed equation must have the same in t, u. This involves that the original equation should involve y, y', y'', &c., only in the forms given in Table III as equivalent to u, u', u'', &c., and should further be homogeneous in those functious reckoned of equal degree.

Ex. $f\left(\dfrac{y''}{y'} - \dfrac{y'}{y}\right) = 0$ is of this kind.

As this depression introduces the independent variable only in place of the original x, if present in the original equation, it is clear that its repeated application does not affect the applicability of depression ii., which can therefore be applied after repeated application of depression iv. if x were originally absent.

Similarly in an equation possessing multiple homogeneity, the repeated application of this depression leaves finally (as in Art. 9) single homogeneity of order -1.

19. *Repeated Depression Formulæ.*

Table VII gives the results of a double application of each of the depressions ii., iii., iv.

Repeated Depression-Formulæ.

Table VII.

x, y the original variables;

x_1, y_1; x_2, y_2 the variables of the first and second depressions.

Repeated Depression ii. *Absence of* x, y.

$$y' = x_2, \quad y'' = x_2 y_2, \quad y''' = x_2 y_2 (x_2 y'_2 + y_2),$$
$$y^{iv} = x_2^3 y_2 (y_2 y_2'' + y_2'^2) + 4x_2^2 y_2^2 y_2' + x_2 y_2^3.$$

Repeated Depression iii. *Homogeneity of order* $\nu = 1$ *in* x, y *and in* x_1, y_1.

$$x \equiv 1, \quad y \equiv 1, \quad y' \equiv x_2 + 1; \quad y'' \equiv x_2 (y_2 + x_2 + 1),$$
$$y''' \equiv x_2^2 y_2 (y_2' + 1) + x_2 (y_2 + x_2)^2 - x_2,$$
$$y^{iv} \equiv x_2^2 y_2 \{x_2 (y_2 y_2'' + y_2'^2 - 1) + 3 (y_2 + x_2)(y_2' + 1)\}$$
$$+ x_2 (x_2 + y_2 - 1)\{x_2 y_2 (y_2 + 1) + (x_2 + y_2)^2 - 1\}.$$

REPEATED DEPRESSION iv. *Homogeneity of order* ∞ *in* x, y *and in* x_1, y_1.

$$x \equiv x_2, \quad y \equiv 1, \quad y' \equiv 1, \quad y'' \equiv y_2 + 1,$$

$$y''' \equiv y_2' + y_2^2 + 3y_2 + 1,$$

$$y^{iv} \equiv y_2'' + 3y_2y_2' + 4y_2' + y_2^3 + 7y_2^2 + 6y_2 + 1.$$

20. *Other depressible cases.* Inasmuch as Depressions ii., iii., iv. introduce in general both the independent and dependent variables (t, u) in place of x, y, y', &c, it may happen that certain functions of x, y, y', &c. in the original give rise on application of those depressions to a depressed equation free from one or other of the new variables (u, t), and therefore depressible by Depressions i. or ii., although the original equation may have contained y or x, and was not therefore depressible by Depressions i. or ii.

21. *Absence of* u, &c. (Depression i.). On examining Table III. it is at once seen that—

"A differential equation wanting x, or homogeneous in any order, and also involving x, y, y', y'', &c. only in the forms equivalent to t, $u^{(r)}$, $u^{(r+1)}$, &c. of the Depression ii, iii, or iv, (as the case may be) will—after that depression—be found free from the r quantities u, u', u'', ...$u^{(r-1)}$, and be therefore further depressible by Depression i."

22. *Absence of* t (Depression ii.). On examining the depression-formulæ of Depression iii. (Table II.), it is seen that t enters in the first degree only into several of the substitutions for y, y', y'', &c., and will therefore disappear from certain simple functions thereof, depending on the value of ν, as follows:—

ν	t present only in	t absent from
0	y	y', y'', y''', &c.
1	y, y'	$(xy' - y)$
2	y, y', y''	$(xy' - 2y)$, $(xy'' - y')$; $(x^2y'' - 2y)$
8	y, y', y'', y'''	$(xy' - 3y)$, $(xy'' - 2y')$, $(xy''' - y'')$; $(x^2y'' - 3.2y)$, $(x^2y''' - 2.1y')$, $(x^3y''' - 3.2.1y)$
...	....	....
ν	y, y', y'' to y^{ν}	$(xy' - \nu y)$, $\{xy'' - (\nu - 1)\, y'\}$, $\{xy''' - (\nu - 2)\, y''\}$, &c. $\{x^2y'' - \nu\, (\nu - 1)\, y\}$, $\{x^2y''' - (\nu - 1)\, (\nu - 2)\, y\}$, &c. ...$\{x^{\nu}y^{(\nu)} - {}^{\nu}\, (\nu - 1)\, (\nu - 2)$, ...$3.2.1.y\}$.

Hence it follows that—

"A differential equation having homogeneity of any order ν (ν not infinite), and either not containing the $(\nu+1)$ quantities $y, y', y'', \ldots y^{(\nu)}$, or else containing them only in the forms shewn above for the value of ν, will after Depression iii. leave an equation free from t, and therefore further depressible by Depression ii."

In the case of an equation with multiple homogeneity, there is thus a large range of applicability, as the value of ν may then be chosen at will.

23. *Preparation of equations.* An equation not possessing the singularities here considered may sometimes be transformed into one possessing them by a suitable change of variables.

Removal of x. The well-known results.

When $x = \varepsilon^{\theta}$,

$$xD_x = D_\theta,$$

$$f(xD_x) = f(D_\theta),$$

$$x^r.D_x^{\,r} = D_\theta(D_\theta - 1)(D_\theta - 2)\ldots\{D_\theta - (r-1)\}.$$

When $a + bx = \varepsilon^{b\theta}$,

$$(a+bx)\,D_x = D_\theta,$$

$$f\{(a+bx)\,D_x\} = f(D_\theta),$$

$$(a+bx)^r\,D_x^{\,r} = D_\theta(D_\theta - b)(D_\theta - 2b)\ldots\{D_\theta - (r-1)\,b\},$$

enable the variable x to be removed from equations which contain x and the differentials only in forms xD_x, $f(xD_x)$, $x^r.D_x^{\,r}$, or only in forms $(a+bx)D_x$, $f\{(a+bx)\,D_x\}$, $(a+bx)^r D_x^{\,r}$.

24. *Integration.* The use of these depressions is chiefly as a help in the integration of high-order non-linear differential equations. The result of the depressions is either a differential equation of lower order or an equation free of differentials (when the order of the original is equal to the number of its singularities): in the former case the depressed differential equation must be integrated.

In either case this equation, free of differentials, is the starting point for a series of ascending steps in which the successive depressions are reversed one by one. Each reversal of a depression gives rise to a first-order differential equation which is to be integrated before passing on to the next Step: these correspond to the depressions for homogeneity and

absence of x. Lastly there will be a series of r simple integrations, corresponding to the depression for absence of $y, y', y'', \ldots y^{(r-1)}$.

25. *Conditions for success.* The conditions for the practical solution of a high-order differential equation by this process are—

1°. The final depressed equation must be either free of differentials, or else a solvible differential equation.

2°. The first-order differential equations arising must be separately solvible.

3°. The final simple integrations (r in number) must be separately possible.

26. *Example.* As an example of these principles, the equation—

$$9y''^2y^{v} - 45y''y'''y^{iv} + 40y'''^3 = 0,$$

which is the well known differential equation of a conic may be taken. It wants x, y, y', and has multiple homogeneity, so may be depressed five orders in all, the result being an algebraic equation.

As shewn in Art. 12, there are three Courses open, viz. the Depressions may be taken in any of three orders as below, (beginning always with Depression i. applied to the utmost extent),

Course	1°	2°	3°
Depression	i., ii., iii. ($\nu=\nu$), iii. ($\nu=1$)	i., ii., iv., iii. ($\nu=-1$)	i., iv., ii., iii. ($\nu=2$)

The results are shewn, step by step, in Table VIII.: the final result (Step IV.) of each Course might have been written down at once by using the formulæ of Table V., without shewing the intermediate steps.

The integration of the three final depressed equations of each Course is shewn in Table IX. It has been thought sufficient to indicate the leading steps without shewing the actual details of the integrations (which would cover several pages).

It will be seen that—in this particular example—it has been possible at each step to solve the first-order differential equations *algebraically*, so as to exhibit the differential coefficients, as explicit functions, upon which the solution can be effected by separation of variables. In particular in Course 1°,

the result of reversal of Step IV. reversed is

$$cx_3^{\frac{1}{3}} = \{y_3 + (\nu - \tfrac{5}{3})\,x_3\}^{\nu-\frac{5}{3}}\,\{y_3 + (\nu - \tfrac{4}{3})\,x_3\}^{-\nu+\frac{4}{3}},$$

and the substitutions for reversing Step III. are

$$x_3 = x_2^{-\nu}y_2, \quad y_3 = x_2\frac{dx_3}{dx_2}, \quad (\nu \text{ being arbitrary}).$$

This gives rise to a first-order differential equation involving dy_2/dx_2 in a complex form, and therefore difficult to solve directly. But inasmuch as the result should be independent of ν, it is permissible to assign such value to ν as will facilitate the solution. It will be found that with any of the values $\nu = \frac{4}{3}, \frac{5}{3}, \frac{3}{2}, 1, 2$ the equation may be algebraically solved so as to exhibit dx_3/dx_2 as an explicit function, upon which the solution can be effected by separation of variables.

Depression of Differential Equation of Conic.

Table VIII.

$$9y''^2y^{v} - 45y''y'''y^{iv} + 40y'''^3 = 0,$$

(an equation wanting x, y, y', and possessing multiple homogeneity).

Step I. *Depression* i. (take $y'' = y_1$, $x = x_1$, so as to depress *two* degrees at once).

$$9y_1^2y_1''' - 45y_1y_1'y_1'' + 40y_1'^3 = 0,$$

(an equation wanting x_1, and possessing multiple homogeneity).

There are now three Courses open, as explained above.

Course 1°. Step II. *Depression* ii.

$$9x_2^2(y_2y_2'' + y_2'^2) - 45x_2y_2y_2' + 40y_2^2 = 0,$$

(an equation possessing multiple homogeneity).

Step III. *Depression* iii. (ν arbitrary).

$$9\left[x_3\{y_3y_3' + (2\nu - 1)\,y_3 + \nu(\nu - 1)\,x_3\} + (y_3 + \nu x_3)^2\right]$$
$$- 45x_3(y_3 + \nu x_3) + 40x_3^2 = 0,$$

(an equation homogeneous in order 1).

COURSE 1°. STEP IV. *Depression* iii. ($\nu = 1$).

$$x_4 y_4 + 2\left(x_4 + \nu - \tfrac{5}{3}\right)\left(x_4 + \nu - \tfrac{4}{3}\right) = 0,$$

(the final depressed equation, wherein ν is still arbitrary).

COURSE 2°. STEP II. *Depression* ii.

$$9x_2^2\left(y_2 y_2'' + y_2'^2\right) - 45x_2 y_2 y_2' + 40y_2^2 = 0,$$

(an equation possessing multiple homogeneity).

STEP III. *Depression* iv.

$$9x_3^2\left(y_3' + 2y_3^2\right) - 45x_3 y_3 + 40 = 0,$$

(an equation homogeneous in order $\nu = -1$).

STEP IV. *Depression* iii.

$$y_4 + 2\left(x_4 - \tfrac{5}{3}\right)\left(x_4 - \tfrac{4}{3}\right) = 0,$$

(the final depressed equation).

COURSE 3°. STEP II. *Depression* iv.

$$9y_2'' - 18y_2 y_2' + 4y_2^3 = 0,$$

(an equation wanting x_2, and homogeneous in order $\nu = -1$).

STEP III. *Depression* ii.

$$9y_3 y_3' - 18x_3 y_3 + 4x_3^3 = 0,$$

(an equation homogeneous in order $\nu = 2$).

STEP IV. *Depression* iii. ($\nu = 2$).

$$x_4 y_4 + 2\left(x_4 - \tfrac{2}{3}\right)\left(x_4 - \tfrac{1}{3}\right) = 0,$$

(the final depressed equation).

DIFFERENTIAL EQUATION OF CONIC.

TABLE IX.

Integration of final depressed equation of Course 1°, 2° or 3°.

COURSE 1°. *Depressions* i., ii., iii. ($\nu = \nu$), *and* iii. ($\nu = 1$).
Final depressed equation.

$$x_4 y_4 + 2\left(x_4 + \nu - \tfrac{5}{3}\right)\left(x_4 + \nu - \tfrac{4}{3}\right) = 0.$$

COURSE 1°. STEP I. *Depression* iii. ($\nu=1$) *reversed.*

Formulæ. $x_4=\dfrac{y_3}{x_3}$, $y_4=x_3\dfrac{dx_4}{dx_3}$.

Equation in x_3, x_4. $cx_3^{\frac{2}{3}}=\dfrac{(x_4+\nu-\frac{5}{3})^{\nu-\frac{5}{3}}}{(x_4+\nu-\frac{4}{3})^{\nu-\frac{4}{3}}}$.

Eqvation in x_3, y_3. $cx_3^{\frac{1}{3}}=\dfrac{\{y_3+(\nu-\frac{5}{3})\,x_3\}^{\nu-\frac{5}{3}}}{\{y_3+(\nu-\frac{4}{3})\,x_3\}^{\nu-\frac{4}{3}}}$.

N.B. In these latter two results ν is arbitrary.

STEP II. *Depression* iii. (ν arbitrary) *reversed.*

Equation in x_3, y_3 *solved for* y_3.

Make $$\nu=\tfrac{4}{3} \text{ or } \tfrac{5}{3},$$

therefore $$y_3=\frac{1}{cx_3}\pm\tfrac{1}{3}x_3.$$

Formulæ. $x_3=x_2^{-\nu}y_2$, $y_3=x_2\dfrac{dx_3}{dx_2}$.

Equation in x_2, x_3. $cx_2^{\pm\frac{2}{3}}=C\pm x_3^2$.

Equation in x_2, y_2. $y_2=x_2^{\frac{3}{2}}\sqrt{(A_2x_2^{\frac{1}{3}}+B_2x_2^{-\frac{1}{3}})}$.

STEP III. *Depression* ii. *reversed.*

Formulæ. $x_2=y_1$, $y_2=\dfrac{dy_1}{dx_1}$.

Equation in x_1, y_1. $y_1=(A_1+B_1x_1+C_1x_1^2)^{-\frac{3}{2}}$.

STEP IV. *Depression* i. *reversed.*

Formulæ. $x_1=x$, $y_1=y''$.

Final equation in x, y. $y=Ax+B\pm\sqrt{(Cx^2+Bx+D)}$.

which is the required complete primitive, and is the algebraic equation of a conic.

COURSE 2°. *Depressions* i., ii., iv., *and* iii. ($\nu=-1$).

Final depressed equation. $y_4+2\,(x_4-\frac{5}{3})\,(x_4-\frac{4}{3})=0$.

STEP I. *Depression* iii. ($\nu = -1$) *reversed.*

Formulæ. $x_4 = x_3 y_3,\ y_4 = x_3 \dfrac{dx_4}{dx_3}$.

Equation in x_3, x_4. $cx_3^{\frac{3}{2}} = \dfrac{x_4 - \frac{4}{3}}{x_4 - \frac{5}{3}}$.

Equation in x_3, y_3. $x_3 y_3 = \frac{1}{6} \dfrac{cx_3^{\frac{3}{2}} + 1}{cx_3^{\frac{3}{2}} - 1} + \frac{3}{2}$.

STEP II. *Depression* iv. *reversed.*

Formulæ. $x_3 = x_2,\ y_3 = \dfrac{1}{y_2} \dfrac{dy_2}{dx_2}$.

Equation in x_2, y_2. $y_2 = x_2^{\frac{3}{2}} \sqrt{(A_2 x_2^{\frac{1}{2}} + B_2 x_2^{-\frac{1}{2}})}$.

STEPS III and IV. Same as in COURSE 1°, *q.v.*

COURSE 3°. *Depressions* i., iv., ii. *and* iii. ($\nu = 2$).

Final depressed equation. $x_4 y_4 + 2 (x_4 - \frac{2}{3})(x_4 - \frac{1}{3}) = 0$.

STEP I. *Depression* iii. ($\nu = 2$) *reversed.*

Formulæ. $x_4 = \dfrac{y_3}{x_3^2},\ y_4 = x_3 \dfrac{dx_4}{dx_3}$.

Equation in x_3, x_4. $cx_3^2 = \dfrac{x_4 - \frac{1}{3}}{(x_4 - \frac{2}{3})^2}$.

Equation in x_3, y_3. $\frac{3}{2} C y_3 = 1 + Cx_3^2 \pm \sqrt{(1 + Cx_3^2)}$.

STEP II. *Depression* ii. *reversed.*

Formulæ. $x_3 = y_2,\ y_3 = \dfrac{dy_2}{dx_2}$.

Equation in x_2, y_2. $y_2 = \dfrac{3 (C_2 x_2 + B_2)}{A_2 + 2B_2 x_2 + C_2 x_2^2}$.

STEP III. *Depression* iv. *reversed.*

Formulæ. $x_2 = x_1,\ y_2 = \dfrac{1}{y_1} \dfrac{dy_1}{dx_1}$.

Equation in x_1, y_1. $y_1 = (A_1 + B_1 x_1 + C_1 x_1^2)^{-\frac{3}{2}}$.

STEP. IV. Same as in COURSE 1°, *q.v.*

CONTENTS.

The following papers have been received:

Mr. Buchheim, "Note on Matrices in Involution."

Prof. Cayley, "Analytical formulæ in regard to an octad of points," "A correspondence of confocal Cartesians with the right lines of a hyperboloid," "Note on the relation between the distances of five points in space."

Prof. Mathews, "Geometry on a quadric surface."

Mr. L. J. Rogers, "An extension of the A. M. and G. M. theorem in inequalities."

Articles for insertion will be received by the Editor, or by Messrs. Metcalfe and Son, Printing Office, Trinity Street, Cambridge.

No. CCII.] NEW SERIES. [February, 1888.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XVII.—NO. 10.

MACMILLAN AND CO.

London and Cambridge.

1888.

Price—One Shilling.

METCALFE AND SON, CAMBRIDGE

It will be seen that, by starting with depression i. *applied to its utmost extent* as Step I., only four *separate* steps of depression have been necessary, and the *ascending* steps (which are the most difficult) have involved the solution of only *three* (first order) differential equations and *two* simple integrations.

DIFFERENTIAL EQUATION OF CONIC. OTHER DEPRESSIONS.

Inasmuch as this differential equation is clear of x, y, y', it may be depressed one order by depression i., and will still be clear of the new variables x, y, and will still be multiply homogeneous. It may therefore be depressed (Art. 16) two orders by a double application of depression ii. without destroying the homogeneity; moreover the depression for homogeneity of order ∞ may be effected before or after the second of depressions ii. (Art. 9).

Hence the 5 depressions required in all may be effected in any of the following orders:—

Course 1°. Depressions i., ii., ii., iii. (ν arbitrary), iii. ($\nu=1$).

„ 2°. „ i., ii., ii., iv., iii. ($\nu=-1$).

„ 3°. „ i., ii., iv., ii., iii. ($\nu=2$).

But this procedure is of course not so easy as that before shewn in abstract in Tables VIII. and IX., especially in the ascending steps, the reversal of Step II. not being so easy as in the former procedure.

It is proposed to show the detail in a further paper.

AN EXTENSION OF A CERTAIN THEOREM IN INEQUALITIES.

By *L. J. Rogers.*

§1. I PROPOSE in the following pages to show how, by a slight extension of the well-known theorem in inequalities concerning the arithmetic and geometrical means of n positive quantities, we can deduce many others, including those usually given in the text-books.

The theorem is as follows:

If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be all positive quantities, then

$$\left(\frac{a_1b_1+a_2b_2+\ldots a_nb_n}{a_1+a_2+\ldots a_n}\right)^{a_1+a_2+\ldots a_n} > b_1^{a_1}b_2^{a_2}\ldots b_n^{a_n} \ldots\ldots(1).$$

Firstly, let $a_1, a_2, \ldots a_n$ be integers.

Then we merely have a particular case of the well-known theorem, wherein we have a_1 quantities, each equal to b_1, &c., the whole number of them being $a_1+a_2+\ldots a_n$.

Secondly, let $a_1, a_2, \ldots$ be fractional.

Let N be the least common measure of their denominators and let $Na_1=A_1$, $Na_2=A_2$, &c., then we get by what is proved above

$$\left(\frac{A_1b_1+A_2b_2+\ldots}{A_1+A_2+\ldots}\right)^{A_1+A_2+\ldots} > b_1^{A_1}b_2^{A_2}\ldots.$$

Taking the real positive N^{th} root of each side we get after reducing the bracketted fraction, the inequality (1).

Thirdly, let the a's be incommensurable.

Then we may substitute for each of these quantities fractions, which may differ from them by less than any assigned quantities, and since the theorem is true for the substituted fractions, we may assume it also true for the given incommensurables.

Hence we may consider (1) as established.

It will be found conveniently brief to write s_r for $a_1^r+a_2^r+\ldots$, as we shall do henceforth.

Let $b_r=\dfrac{1}{a_r}$ for all values of r from 1 to n, then from (1)

we get $$\left(\frac{s_0}{s_1}\right)^{s_1} > \frac{1}{a_1^{a_1}}\frac{1}{a_2^{a_2}}\ldots,$$

i.e. $$\left(\frac{a_1+a_2+\ldots a_n}{n}\right)^{a_1+a_2+\ldots a_n} < a_1^{a_1}a_2^{a_2} \ldots\ldots\ldots\ldots (2),$$

a well known result.

Write a_1^r for a_1, a_2^r for a_2, &c., and let $b_1=a_1^{m-r}$, $b_2=a_2^{m-r}$, where $m>r$. Then (1) gives

$$\left(\frac{s_m}{s_r}\right)^{s_r} > (a_1^{a_1^r}a_2^{a_2^r}\ldots)^{m-r}.$$

Again, let $b_1=a_1^{t-r}$ where $t<r$.

Then $$\left(\frac{s_t}{s_r}\right)^{s_r} > (a_1^{a_1^r}a_2^{a_2^r}\ldots)^{-(r-t)},$$

i.e. $$\left(\frac{s_r}{s_t}\right)^{s_r} < (a_1^{a_1^r}\ldots)^{r-t}.$$

Combining these results we get

$$\left(\frac{s_m}{s_r}\right)^{\frac{s_r}{m-r}} > \left(\frac{s_r}{s_t}\right)^{\frac{s_r}{r-t}}.$$

Taking the s_rth root and reducing we get

$$s_m^{r-t} s_t^{m-r} > s_r^{m-t} \quad \text{......................} \quad (3),$$

provided $\qquad m > r > t.$

Let $t = 0$.

Then $$\left(\frac{s_m}{s_0}\right)^r > \left(\frac{s_r}{s_0}\right)^m \quad \text{......................} \quad (4),$$

which for $r = 1$, so that $m > 1$, we get

$$\frac{a_1^m + a_2^m + \dots a_n^m}{n} > \left(\frac{a_1 + a_2 + \dots a_n}{n}\right)^m,$$

a well known relation, if $m > 1$.

If $m = 1$, $t = 0$, then $\left(\frac{s_1}{s_0}\right)^r > \frac{s_1}{s_0}$, where r is a positive proper fraction.

If $m = 0$, $t = -1$, $\qquad \left(\frac{s_{-1}}{s_0}\right)^r < \frac{s_{-1}}{s_0}$,

and if $m = 0$, $r = -1$, then $\left(\frac{s_{-1}}{s_0}\right)^t < \frac{s_{-1}}{s_0}$.

Again, from (3) we may deduce many other results.

If $t > u$, we also have

$$s_r^{t-u} s_m^{r-t} > s_t^{r-u},$$

and if $m + u = r + t$, we get on multiplication

$$(s_m s_u)^{r-t} > (s_r s_t)^{r-t},$$

or $$s_m s_u > s_r s_t \quad \text{.........................} \quad (5),$$

provided $m > r > t > u$, and $m + u = r + t$.

The result (5) can be obtained without using (1).

If we multiply the two arrays,

$$\begin{vmatrix} a_1^{t-u}, & a_2^{t-u}, & \dots, & a_n^{t-u} \\ 1, & 1, & \dots, & 1 \end{vmatrix}$$

and $$\begin{vmatrix} a_1^r, & a_2^r, & \dots, & a_n^r \\ a_1^t, & a_2^t, & \dots, & a_n^t \end{vmatrix},$$

we get the relation

$$\begin{vmatrix} s_m, & s_r \\ s_t, & s_u \end{vmatrix} = \Sigma a_1^t a_2^t (a_1^{t-u} - a_2^{t-u})(a_1^{r-t} - a_2^{r-t}),$$

the right side of which contains only positive terms.

Therefore $$s_m s_u > s_r s_t.$$

L2

Let $u=0$ in (5), so that $s_{r+t}s_0 > s_r s_t$, we see then that in the same way

$$s_{\alpha+\beta+\gamma}s_0 > s_\alpha s_{\beta+\gamma},$$

$$s_{\alpha+\beta+\gamma}s_0^2 > s_\alpha s_{\beta+\gamma}s_0 > s_\alpha s_\beta s_\gamma,$$

and so on, or as we may better write it,

$$\frac{s_{\alpha+\beta+\gamma}}{s_0^3} > \frac{s_\alpha}{s_0}\frac{s_\beta}{s_0}\frac{s_\gamma}{s_0} \text{ .... } (6),$$

a result which admits of easy extension to n suffixes $\alpha, \beta, \gamma, \dots$.

We shall now pass on to applications of the above results to Integral Calculus.

§ 2. In the inequality § 1 (3) let

$$a_1 = f(x_1+h), \quad a_2 = f(x_1+2h), \quad \dots, \quad a_n = f(x_1+nh),$$

where $$x_1 + nh = x_2.$$

We then get, after multipyling s_m, s_r, s_t by h, and putting $f(x)=y$, and making h decrease indefinitely,

$$\{\int y^m dx\}^{r-t}\{\int y^t dx\}^{m-r} > \{\int y^r dx\}^{m-t} \text{............} (1),$$

where $m>r>t$, and the limits are such that y^m, y^r, and y^t remain finite and positive for all values between these limits.

As an example of this we may put $y\equiv x$, whence after changing $m+1$ to m, &c., it follows that

$$\left(\frac{r}{t}\right)^m \left(\frac{t}{m}\right)^r \left(\frac{m}{t}\right)^t > 1 \text{ .... } (2),$$

where $m>r>t$.

Here we take for limits 1 and 0.

From (1) we may observe that it is impossible that

$$\int_b^a u^m dx \times \int_{b'}^{a'} v^m dx = 1 \text{ .... } (3),$$

where the limits are independent of m and taken so that the functions u, v should remain positive between their respective limits.

For let $$\int_b^a u^m dx = \phi(m).$$

Then, by (1),

$$\{\phi(m)\}^{r-t}\{\phi(t)\}^{m-r} > \{\phi(r)\}^{m-t}.$$

But if (3) were true we should have the reverse inequality also true, which is impossible.

Hence (3) cannot hold good.

As an example we have

$$\int_0^1 x^m dx = \frac{1}{m+1},$$

so that no function u can be found such that

$$\int_b^a u^m dx = m + 1,$$

where a, b are subject to the afore-stated conditions.

As we deduced (1) from § 1 (3), so may we draw from § 1 (4) that

$$(x_1 - x_2)^{m-r} \left\{\int_{x_2}^{x_1} y^m dx\right\}^r > \left\{\int_{x_2}^{x_1} y^r dx\right\}^m \text{ ....(4)},$$

where $m > r$, and the limits are under conditions as before.

If $y \equiv x$, we get

$$(r+1)^m > (m+1)^r \text{ ....(5)},$$

where $$m > r.$$

§ 3. If we treat the inequality § 1 (3) in the same way as we deduced § 1 (1) from the well-known A.M. and G.M. relation, we shall get

$$(\Sigma ab^m)^{r-t} (\Sigma ab^t)^{m-r} > (\Sigma ab^r)^{m-t} \text{ ....(1)}.$$

From § 1 (5) we get

$$\Sigma ab^m . \Sigma ab^n > \Sigma ab^r . \Sigma ab^t \text{ ....(2)},$$

and from § 1 (6)

$$\frac{\Sigma ab^{\alpha+\beta+\gamma+\dots}}{\Sigma a} > \frac{\Sigma ab^\alpha}{\Sigma a} \frac{\Sigma ab^\beta}{\Sigma b} \text{ ....(3)}.$$

These give results similar to § 2 (1), viz.

$$(\int yv^m dx)^{r-t} (\int yv^t dx)^{m-r} > (\int yv^r dx)^{m-t} \text{ ....(4)},$$

$$\int yv^m dx \int yv^n dx > \int yv^r dx \int yv^t dx \text{ ....(5)},$$

$$\frac{\int yv^{\alpha+\beta+\dots} dx}{\int y\, dx} > \frac{\int yv^\alpha dx}{\int y\, dx} \frac{\int yv^\beta dx}{\int y\, dx} \text{ ....(6)},$$

where y, v are functions of x, and the limits are under the same conditions as before.

In (4) put $y = \varepsilon^{-x}$, $v = x$, then

$$(\Gamma m)^{r-t}(\Gamma t)^{m-r} > (\Gamma r)^{m-t}.$$

If $y = x^{l-1}$, $v = 1 - x$,

$$B(l.m)^{r-t} B(l.t)^{m-r} > B(l.r)^{m-t}.$$

If $u = \dfrac{1}{1+x^q}$, $v = x$,

$$\sin^{r-t}\frac{m\pi}{q}\sin^{m-r}\frac{t\pi}{q} < \sin^{m-t}\frac{r\pi}{q},$$

provided the sines are all positive.

We may also get similar inequalities from (5) and (6).

As in § 2 (3), we may also show that if

$$\int_b^a yv^m dx = \phi(m),$$

where a, b, y, v are subject to the same conditions, then we cannot have

$$\int_{b'}^{a'} uz^m dx = \frac{1}{\phi(m)} \qquad \text{(7)},$$

with conditions as before.

The following equations are therefore absurd:

$$\int_{x_2}^{x_1} uz^m dx = \frac{1}{\Gamma m}, \text{ or } \frac{1}{B(l, m)}, \text{ or } \sin m\pi,$$

if uz^m is always positive between $x = x_1$ and $x = x_2$, and x_1, x_2 are independent of m.

§ 4. We may also obtain a few inequalities from taking logarithms in § 1 (1), whence

$$\log\frac{\Sigma ab}{\Sigma a} > \frac{\Sigma a \log b}{\Sigma a}.$$

From this may be deduced

$$\log\frac{\int vy\,dx}{\int v\,dx} > \frac{\int v \log y\,dx}{\int v\,dx},$$

with restrictions as before as to limits.

These last inequalities do not appear to lead to very interesting results.

Oxford, *Nov.* 1, 1887.

GEOMETRY ON A QUADRIC SURFACE.

By Prof. *Mathews.*

From a fixed point P of a quadric surface project plane sections of the surface upon a plane which is parallel to the tangent plane at P; then the projections will all be similar and similarly situated.

This is easily proved analytically; or geometrically, by straining the stereographic projection of a sphere.

In particular suppose that P is an umbilic U, then the projections become circles, reducing to straight lines for sections passing through U, and conversely to every straight line or circle in the plane of projection corresponds a plane section of the surface.

To a coaxal system of circles corresponds the system of conics in which the quadric surface S is intersected by an axial pencil of planes. If the axis of the pencil meets S in two real points A and B every conic of the coaxal system will go through A and B; if, on the other hand, A and B are imaginary, no two conics of the coaxal system will intersect in real points. In this latter case there are two "point-conics" of the system, viz. the points of contact of the two planes of the pencil which touch S.

The plane of the pencil which passes through U meets S in a conic, which may be called the radical axis of the system.

The axis of the pencil of planes corresponding to a coaxal system may be called the "polar axis" of the system.

Let ω be the tangent plane to S at the point U, and let g be any straight line in this plane. Then the coaxal system of which g is the polar axis projects into a system of circles having their radical axis at infinity; that is, a concentric system.

Hence we get a construction for what may be called the "centroid" of any conic drawn upon S. Namely, produce the plane of the conic to meet ω in a straight line g; then the point of contact of the other tangent plane to S, which can be drawn through g, is the centroid in question. Or again, join U to the pole of the section; this line will meet S in the required centroid.

The centroids of a coaxal system lie on a conic passing through U.

Let g be the polar axis of a coaxal system; then, if g' be conjugate to g with regard to S, the coaxal system of which g' is the polar axis may be called conjugate to the other system. The centroids of each system lie on the radical axis of the other.

Any two conics upon S determine two points corresponding to the centres of similitude of the circles into which the conics project. They may be found as follows. Find the centroids of the two conics and through them draw two planes intersecting ω (the tangent plane to S at U) in the same straight line. Suppose these planes meet the given conics in A, B and C, D respectively; then, if the conics UAC, UBD meet in O and UAD, UBC in O', O, O' are the points required.

It is needless to go into further detail, for it is evident that every known theorem in the geometry of the straight line and circle gives a corresponding theorem in the geometry of conics upon S. For example, two conics P and Q upon S may be such that a finite number of other conics may be drawn upon S, each touching P and Q and two adjacent conics of the series; and this can be done, if at all, in an infinite number of ways, &c.

Or again, as an example of metrical theorems, let A, B be any two fixed points upon S, and U an umbilic. Draw two conics upon S passing through U, A and U, B respectively, and intersecting at a given angle in U; then the locus of their other point of intersection is a conic passing through A and B.

Again, to a conic in the plane of projection corresponds upon S a quartic having a conjugate point at U; and we have upon S a kind of projective geometry by which any such curve may be derived from a plane section of S, and its properties investigated.

EXPRESSIONS FOR $\Theta(x)$ AS A DEFINITE INTEGRAL.

By *J. W. L. Glaisher*.

In Vol. v. (1876), p. 173 of the *Messenger*, I gave without proof an expression for $\Theta(x)$ as a definite integral. This expression contained several errors which were corrected in a paper in Vol. III. (1887), pp. 61–66 of the *Proceedings* of the Cambridge Philosophical Society. This paper contained six expressions for the function $\Theta(x)$, with an explanation of the

manner in which they were obtained. I have recently re-calculated the values of four of these expressions, which are more symmetrical than the other two. The resulting formulæ are given below, arranged in a somewhat more convenient form. The letter μ denotes $\dfrac{\pi K'}{K}$.

$$\Theta\left(\frac{2Kx}{\pi}\right) = \int_0^\infty \left\{\frac{\sinh\sqrt{(\pi\mu)}\,t - \sin\{\sqrt{(\pi\mu)}\,t+2x\}}{\cosh\sqrt{(\pi\mu)}\,t+\cos\{\sqrt{(\pi\mu)}\,t+2x\}} + \frac{\sinh\sqrt{(\pi\mu)}\,t - \sin\{\sqrt{(\pi\mu)}\,t-2x\}}{\cosh\sqrt{(\pi\mu)}\,t+\cos\{\sqrt{(\pi\mu)}\,t-2x\}}\right\}\sin(\tfrac{1}{2}\pi t^2)\,dt$$

$$= \int_0^\infty \left\{\frac{\sinh\sqrt{(\pi\mu)}\,t + \sin\{\sqrt{(\pi\mu)}\,t+2x\}}{\cosh\sqrt{(\pi\mu)}\,t+\cos\{\sqrt{(\pi\mu)}\,t+2x\}} + \frac{\sinh\sqrt{(\pi\mu)}\,t + \sin\{\sqrt{(\pi\mu)}\,t-2x\}}{\cosh\sqrt{(\pi\mu)}\,t+\cos\{\sqrt{(\pi\mu)}\,t-2x\}}\right\}\cos(\tfrac{1}{2}\pi t^2)\,dt$$

$$= 2\int_0^\infty \frac{P+Q}{\cosh\pi t - \cos\pi t}\sin(\tfrac{1}{2}\mu t^2)\,dt$$

$$= 1 + 2\int_0^\infty \frac{P-Q}{\cosh\pi t - \cos\pi t}\cos(\tfrac{1}{2}\mu t^2)\,dt,$$

where

$$P = \sinh(\tfrac{1}{2}\pi + x)\,t\cos(\tfrac{1}{2}\pi - x)\,t + \sinh(\tfrac{1}{2}\pi - x)\,t\cos(\tfrac{1}{2}\pi + x)\,t,$$

$$Q = \cosh(\tfrac{1}{2}\pi + x)\,t\sin(\tfrac{1}{2}\pi - x)\,t + \cosh(\tfrac{1}{2}\pi - x)\,t\sin(\tfrac{1}{2}\pi + x)\,t.$$

We may also express the values of P and Q in the form

$$P = 2(\sinh\tfrac{1}{2}\pi t\cos\tfrac{1}{2}\pi t\cosh xt\cos xt + \cosh\tfrac{1}{2}\pi t\sin\tfrac{1}{2}\pi t\sinh xt\sin xt),$$

$$Q = 2(\cosh\tfrac{1}{2}\pi t\sin\tfrac{1}{2}\pi t\cosh xt\cos xt - \sinh\tfrac{1}{2}\pi t\cos\tfrac{1}{2}\pi t\sinh xt\sin xt).$$

In the first two expressions for $\Theta(x)$ the argument x is unrestricted in value; in the last two x must not exceed the limits $\pm\tfrac{1}{2}\pi$.*

* The complete system of expressions, corresponding to those given in this note, for the four Theta functions were communicated to the Cambridge Philosophical Society at the meeting on November 28, 1887.

HOMOGRAPHIC INVARIANTS AND QUOTIENT DERIVATIVES.

By *A. R. Forsyth.*

1. The investigations contained in the present paper were originally begun with the purpose of finding the relation, in which a class of functions called quotient derivatives stand to reciprocants. These quotient derivatives have, among other properties, that of being covariantive for homographic transformations of the dependent and the independent variables, when applied simultaneously and therefore also when applied separately. But it is evident from their forms that their aggregate does not constitute the complete series of such functions; and my first aim has been to obtain these complete series for each of the combinations of homographic transformation.

The character of the invariance is less general than that of M. Halphen's Differential Invariants, which reproduce themselves, save as to a factor, by the substitutions

$$\frac{X}{ax+by+c}=\frac{Y}{a'x+b'y+c'}=\frac{1}{a''x+b''y+c''}.$$

Functions of the kinds herein considered have been previously suggested by Mr. L. J. Rogers,* who has, except in the case of the first kind, limited his investigations to the deduction of the partial differential equations which are satisfied by the functions. His aim was the derivation of homographic reciprocants.

It is by a comparison of the quotient derivatives with these homographic reciprocants that the desired relation has been obtained. The cubic derivative has been expressed in terms of them, but the combination is not legitimate† for the preservation of reciprocal invariance. The relation thus suggested is proved to be general.

2. Perhaps the simplest way of obtaining the quotient derivatives is as follows. Without reproducing the general investigation in which they arise, consider for example a

* "Homographic and Circular Reciprocants" (first paper), *Proc. Lond. Math. Soc.*, vol. XVII. (1886), pp. 220—231.

† Sylvester, "Lectures on the Theory of Reciprocants," *Amer. Journ. Math.*, vol. VIII. (1886), p. 212.

cubic equation

$$\frac{d^3u}{dx^3}=0,$$

of which the primitive is

$$u=A+Bx+Cx^2,$$

and let y be the quotient of two different solutions u_1 and u_2 of this equation, so that

$$yu_1=u_2.$$

Since the third and every higher derivative of u vanish, we have

$$\begin{aligned} u_1y''' + 3y''u_1' + 3y'u_1'' &= 0,\\ u_1y^{iv} + 4y'''u_1' + 6y''u_1'' &= 0,\\ u_1y^{v} + 5y^{iv}u_1' + 10y'''u_1'' &= 0; \end{aligned}$$

and therefore eliminating u_1, u_1', u_1'', which are linearly independent of one another, we have (now indicating differentiation by subscript integers)

$$[y, x]_3 = \begin{vmatrix} y_3, & 3y_2, & 3y_1 \\ y_4, & 4y_3, & 6y_2 \\ y_5, & 5y_4, & 10y_3 \end{vmatrix} = 0.$$

This is a differential equation of the fifth order; its primitive is

$$y=\frac{A+Bx+Cx^2}{D+Ex+Fx^2}.$$

The function on the left-hand side is called the cubic quotient derivative.

The quadratic equation $\dfrac{d^2u}{dx^2}=0$ leads similarly to the well-known Schwarzian

$$[y, x]_2 = \begin{vmatrix} y_2, & 2y_1 \\ y_3, & 3y_2 \end{vmatrix};$$

the quartic equation $\dfrac{d^4u}{dx^4}=0$ leads to a similarly formed quartic quotient derivative

$$[y, x]_4 = \begin{vmatrix} y_4, & 4y_3, & 6y_2, & 4y_1 \\ y_5, & 5y_4, & 10y_3, & 10y_2 \\ y_6, & 6y_5, & 15y_4, & 20y_3 \\ y_7, & 7y_6, & 21y_5, & 35y_4 \end{vmatrix},$$

and so on. And the property of homographic invariance already referred to is constituted by the equation

$$\left[\frac{ay+b}{cy+d}, \frac{ex+f}{gx+h}\right]_n = \frac{(ad-bc)^n}{(eh-fg)^{n^2}} \frac{(gx+h)^{2n^2}}{(cy+d)^{2n}} [y, x]_n.*$$

Some other properties will be obtained later, one of them in particular (§ 45) shewing why in these derivatives the highest differential coefficients of y which occur are only those defined by alternate integers.

Invariants when the independent variable is subject to homographic transformation.

3. Let $\phi(y, x)$ be a function which, when the independent variable x is transformed, reproduces itself, save as to a power of $\frac{dz}{dx}$, so that we may write

$$\phi(y, x) = \left(\frac{dz}{dx}\right)^m \phi(y, z).$$

The exponent m is called the *index* of the invariant.

4. Then the following properties of the general forms of such functions are easily derived :†

(i). *The independent variable does not explicitly occur.* For $x = z + c$, where c is an arbitrary constant, is a possible transformation; if x occurs on the left-hand side of the above invariant equation, there will, after the substitution for x, arise a term or terms in c, which do not occur on the right-hand side.

(ii). *The irreducible invariants do not explicitly contain the dependent variable.* For, if a given invariant ϕ contain y, it can be arranged in powers of y in the form

$$\phi = \phi_0 + y\phi_1 + y^2\phi_2 + \dots.$$

If the transformed value of ϕ be Φ, then we may write

$$\Phi = \Phi_0 + y\Phi_1 + y^2\Phi_2 + \dots;$$

* *Proc. Royal Soc.*, 12th Jan., 1888.

† These correspond to the propositions given by Halphen, Thèse "Sur les invariants différentiels," p. 21.

and since ϕ is an invariant, we have

$$\phi=\left(\frac{dz}{dx}\right)^m \Phi.$$

Hence $\quad \phi_0-\left(\frac{dz}{dx}\right)^m \Phi_0+y\left\{\phi_1-\left(\frac{dz}{dx}\right)^m \Phi_1\right\}+\ldots=0.$

Now ϕ_0 and Φ_0 do not explicitly contain y, which is subject to no variation; hence

$$\phi_0-\left(\frac{dz}{dx}\right)^m \Phi_0=0.$$

Similarly $\qquad \phi_1-\left(\frac{dz}{dx}\right)^m \Phi_1=0,$

and so on; from which it follows that $\phi_0, \phi_1, \phi_2, \ldots$ are invariants of index m. And they are all explicitly free from the dependent variable.

(iii) *Invariants are of uniform grade*, that is, the sum of the orders of differentiation of the dependent variable in the factors of any term of an invariant is the same for all the terms of that invariant. For a possible transformation is $z=ax$, where a is a constant; and then

$$\frac{d}{dx}=a\frac{d}{dz},$$

so that the effect of the transformation on any term is to multiply it by a power of a equal to the grade of the term. But the effect of this transformation on the invariant is to multiply it by a^m, whence the result.

From this it at once follows that the *index of an invariant of the class at present considered is equal to its grade.*

(iv). *Irreducible invariants are homogeneous in the differential coefficients of the dependent variable.* For let an invariant ψ of index m be arranged in the form

$$\phi_\lambda+\phi_\mu+\phi_\nu+\ldots,$$

where ϕ_ρ is the aggregate of terms in ψ of the degree ρ; then we have

$$\phi_\lambda+\phi_\mu+\ldots=\left(\frac{dz}{dx}\right)^m \{\Phi_\lambda+\Phi_\mu+\ldots\}.$$

Now a change of the independent variable in any differential coefficient of y gives a quantity which is linear in

the new differential coefficients; hence the degree of a term in the invariant is unaltered. The last equation therefore shews that the aggregate of terms of any degree transforms into a corresponding aggregate of the same degree, and that this aggregate is an invariant.

It is evident that all these results hold for any transformation of the independent variable.

5. Now for any transformation the general law of differentiation is

$$\frac{1}{r!}\frac{d^r}{dx^r}=\sum_{s=1}^{s=r}\frac{1}{s!}C_{r,s}\frac{d^s}{dz^s},$$

where

$$C_{r,s}=\text{coefficient of } \rho^r \text{ in } \left\{\rho z_1+\frac{1}{2!}\rho^2 z_2+\frac{1}{3!}\rho^3 z_3+\dots\right\}^s,$$

and
$$z_1=\frac{dz}{dx},\ z_2=\frac{d^2z}{dx^2},\ \dots .$$

One method of obtaining the characteristic differential equations, satisfied by the invariants, is to consider the effect on the invariant equation consequent on an arbitrary infinitesimal change in the independent variable. Such a change may be taken in the form

$$z=x+\varepsilon\mu,$$

where ε is an infinitesimal and constant, and μ is an arbitrary function of x; we then have

$$z_1=1+\varepsilon\mu_1,$$

and, for $r>1$,

$$z_r=\varepsilon\mu_r.$$

Hence we deduce

$$C_{r,r}=1+r\varepsilon\mu_1,$$

and, for $s<r$,

$$C_{r,s}=\frac{s\varepsilon}{r-s+1!}\mu_{r-s+1};$$

and therefore

$$\frac{d^r}{dx^r}-\frac{d^r}{dz^r}=r\varepsilon\mu_1\frac{d^r}{dz^r}+\varepsilon\mu_2\frac{r!}{r-2!\,2!}\frac{d^{r-1}}{dz^{r-1}}+\varepsilon\mu_3\frac{r!}{r-3!\,3!}\frac{d^{r-2}}{dz^{r-2}}+\dots$$

$$\dots+\varepsilon\mu_r\frac{r!}{0!\,r!}\frac{d}{dz}$$

$$=r\varepsilon\mu_1\frac{d^r}{dx^r}+\varepsilon\mu_2\frac{r!}{r-2!\,2!}\frac{d^{r-1}}{dx^{r-1}}+\dots+\varepsilon\mu_r\frac{d}{dx},$$

when quantities of only the first order are retained.

6. In the case of a homographic transformation, say

$$z = \frac{ex + f}{gx + h},$$

so that

$$z_1 = \frac{eh - fg}{(gx + h)^2},$$

it is sufficient, for reduction to what precedes, to take $e = h$, $f = 0$, $g = -\frac{1}{2}\varepsilon$; and then

$$z_1 = 1 + \varepsilon x,$$

so that

$$\mu_1 = x,$$

$$\mu_2 = 1,$$

and higher differential coefficients of μ vanish. The preceding general formula now becomes

$$\frac{d^r}{dx^r} - \frac{d^r}{dz^r} = r\varepsilon x \frac{d^r}{dx^r} + \tfrac{1}{2}r(r-1)\varepsilon \frac{d^{r-1}}{dx^{r-1}}.$$

Applying this to the equation

$$\phi(y, x) = z_1^m \phi(y, z)$$
$$= (1 + m\varepsilon x)\phi(y, z),$$

we have on substituting for the differential coefficients of y on the left-hand side an equation

$$\phi(y, z) + \sum_{r=1}\left(r\varepsilon x y_r \frac{\partial\phi}{\partial y_r}\right) + \tfrac{1}{2}\varepsilon\Sigma r(r-1)\, y_{r-1}\frac{\partial\phi}{dy_r}$$
$$= (1 + m\varepsilon x)\phi(y, z),$$

where in terms multiplied by ε we may take z or x indifferently as the variable. Since this equation is to be identically true, we have

$$\sum_{r=1}\left(r y_r \frac{\partial\phi}{dy_r}\right) = m\phi,$$

an index-equation, or say the *grade-equation*; and

$$\Sigma r(r-1)\, y_{r-1}\frac{d\phi}{dy_r} = 0,$$

a *form-equation*.

When the form of the function is determined by the latter and the index is inferred from the form, then the grade equation is identically satisfied.

7. These equations coincide with the equations otherwise obtained by Mr. Rogers (l.c.); and, as he has given a succession of functions satisfying them, it is not proposed here to do more than merely to state results.

An important proposition, which admits of easy proof and will be subsequently used, is the following:

If ϕ be an invariant of grade-index m, then $y_1 \dfrac{d\phi}{dx} - my_2\phi$ is an invariant of grade-index $m+2$.

For
$$\frac{\phi(y, x)}{\left(\dfrac{dy}{dx}\right)^m} = \frac{\phi(y, z)}{\left(\dfrac{dy}{dz}\right)^m},$$

so that
$$\frac{d}{dx}\left(\frac{\phi}{y_1^m}\right) = z_1 \frac{d}{dz}\left\{\frac{\phi(y, z)}{\left(\dfrac{dy}{dz}\right)^m}\right\},$$

or $\dfrac{d}{dx}(\phi y_1^{-m})$ is an invariant of grade-index unity, and therefore $y_1 \dfrac{d\phi}{dx} - my_2\phi$ is an invariant of grade-index $m+2$.

It is by the use of this proposition that Mr. Rogers obtains his succession of educed functions, beginning with the customary Schwarzian $y_1y_3 - \frac{3}{2}y_2^2$; from and after the second educt, however, simpler forms can be given as follows.

8. The function of the first degree is
$$f_1 = y_1.$$

The functions of the second degree are:
$$f_3 = y_1y_3 - \tfrac{3}{2}y_2^2,$$
$$f_5 = y_1y_5 - 10y_2y_4 + 10y_3^2,$$
$$f_7 = y_1y_7 - 21y_2y_6 + 105y_3y_5 - \tfrac{175}{2}y_4^2,$$

and, generally,
$$f_{2p+1} = \sum_{s=0}^{s=p}\left\{\epsilon_s y_{s+1} y_{2p-s+1} \frac{2p+1\,!\;2p\,!}{2p-s+1\,!\;2p-s\,!\;s+1\,!\;s\,!}\right\},$$

where
$$\epsilon_p = \tfrac{1}{2}(-1)^p,$$

and for $s = 0, 1, 2, \ldots, p-1$,
$$\epsilon_s = (-1)^s.$$

CONTENTS.

The following papers have been received:

Mr. Buchheim, "Note on matrices in involution."

Prof. Cayley, "Analytical formulæ in regard to an octad of points,' "A correspondence of confocal Cartesians with the right lines of a hyperboloid," "Note on the relation between the distances of five points in space."

Articles for insertion will be received by the Editor, or by Messrs. Metcalfe and Son, Printing Office, Trinity Street, Cambridge.

No. CCIII.] NEW SERIES. [March, 1888.

THE

MESSENGER OF MATHEMATICS.

EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. XVII.—NO. 11.

MACMILLAN AND CO.

London and Cambridge.

1888.

Price—One Shilling.

METCALFE AND SON, CAMBRIDGE